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Spin-Flavor Structure of Large N_c Baryons

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The spin-flavor structure of large N_c baryons is described in the $1/N_c$ expansion of QCD using quark operators. The complete set of quark operator identities is obtained, and used to derive an operator reduction rule which simplifies the $1/N_c$ expansion. The operator reduction rule is applied to the axial currents, masses, magnetic moments and hyperon non-leptonic decay amplitudes in the $SU(3)$ limit, to first order in $SU(3)$ breaking, and without assuming $SU(3)$ symmetry. The connection between the Skyrme and quark representations is discussed. An explicit formula is given for the quark model operators in terms of the Skyrme model operators to all orders in $1/N_c$ for the two flavor case.

I. INTRODUCTION

The properties of baryons have recently been studied in QCD in a systematic expansion in $1/N_c$, where N_c is the number of colors [1,2,3,4]. In the limit $N_c \to \infty$, it has been shown that the baryon sector of QCD has an exact contracted $SU(2F)$ spin-flavor symmetry, where F is the number of light quark flavors [1,5]. This contracted spin-flavor symmetry follows from consistency conditions on meson-baryon scattering amplitudes which must be satisfied for the theory to be unitary. The spin-flavor structure of baryons for finite N_c is given by studying $1/N_c$ corrections to the large N_c limit. The consistency conditions severely restrict the form of subleading $1/N_c$ corrections, so definite predictions can be made at subleading orders. The $1/N_c$ expansion has been used to obtain results for baryon axial vector currents and magnetic moments up to corrections of relative order $1/N_c^2$ and for baryon masses up to relative order $1/N_c^3$ for two and three light quark flavors. Salient results include the vanishing of $1/N_c$ corrections to pion-baryon coupling ratios in a given strangeness sector and an equal spacing rule for decuplet $\to$ octet pion couplings in different strangeness sectors. For the case of three flavors, additional results have been obtained which do not assume $SU(3)$ flavor symmetry and are therefore valid to all orders in $SU(3)$ symmetry breaking. These results give insight into the structure of flavor $SU(3)$ breaking in the baryon sector. The predictions of the $1/N_c$ expansion for baryons are in good agreement with experiment, and explain the phenomenological success of spin-flavor symmetry for the baryon sector of QCD.

There are two natural approaches to the study of the spin-flavor algebra of baryons for large N_c. One can solve the consistency conditions by constructing irreducible representations of the $N_c \to \infty$ contracted $SU(2F)$ spin-flavor symmetry [3]. These irreducible baryon representations are constructed using standard techniques from the theory of induced representations, and are very closely related to the collective coordinate quantization of the Skyrmion in the Skyrme model [6,7,8]. One can also construct solutions to the consistency conditions using quark operators, an approach which is closely related to the non-relativistic quark model. The two methods are equivalent, since the non-relativistic quark model and Skyrme model are identical in the $N_c \to \infty$ limit [9]. The quark model approach was discussed in detail in refs. [10,11], and used to derive results for large N_c baryons. One nice feature of the quark approach is that it is closely tied to the intuitive picture of baryons as quark bound states, and the $1/N_c$ counting is simply related to quark Feynman diagrams. This connection is obvious when the quarks are heavy.

The Skyrme and non-relativistic quark model realizations of the large N_c spin-flavor algebra for baryons in QCD are identical in the $N_c \to \infty$ limit. At finite N_c, the Skyrme and quark representations differ in their organization of $1/N_c$ corrections, but they give equivalent results at a given order in $1/N_c$. In the Skyrme representation, the contracted spin-flavor algebra is realized exactly, which implies that the irreducible baryon representations of the contracted algebra are infinite dimensional. In contrast, the quark representation uses the non-relativistic quark model algebra for finite N_c. The baryon spectrum, in this case, is finite; it consists of a tower of baryon states that terminates at spin $N_c/2$. The Skyrme and quark representations both give rise to operator identities which eliminate redundant operators at

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a given order in the $1/N_c$ expansion. These identities are much simpler in the Skyrme representation than in the quark representation for two flavors [3]. (In particular, some of the operator identites used in the original analysis are not obvious in the quark description, and have not been derived using this method. The derivation is supplied in this work.) However, both the Skyrme and quark representations become quite complicated for more than two flavors. There has been no derivation of all the non-trivial operator identities for the case of three flavors using either method. These operator identities are required for a systematic analysis of the $1/N_c$ expansion for baryons in the three flavor case.

In this paper, we study the spin-flavor structure of baryons for an arbitrary number of colors and flavors. We present a brief review of the quark model representation in Section II, and the $1/N_c$ counting rules for baryon operators in Section III. All the baryon operators are classified using the $SU(2F)$ spin-flavor symmetry group in Section IV. All the non-trivial operator identities among the baryon operators are derived in Section V. The set of independent operators and the operator identities have an elegant group-theoretic classification. The operator analysis of Sections IV–V is done for the general case of F quark flavors. The special cases of two and three light flavors which are of principal physical interest are considered explicitly in Section VI. The operator identities are used to derive a simple operator reduction rule in Section VII, which gives the linearly independent baryon operators at a given order in the $1/N_c$ expansion. The operator analysis for the case of baryons with three flavors without any assumption of $SU(3)$ symmetry is given in Section VIII. The operator analysis is then used to study the static baryon properties (axial currents, masses, magnetic moments, and hyperon non-leptonic decays) in the $SU(3)$ limit, to first order in perturbative $SU(3)$ breaking, and for completely broken $SU(3)$ flavor symmetry in Sections IX–XII. Readers not interested in some of the details can skip Sections IV–VII, and refer only to the identities (for three flavors) in Tables VIII, XI and XII and the operator reduction rules in Sections VII and VIII before proceeding to the discussion of the static baryon properties. Additional group theory required in the analysis is given in the appendices. We reproduce some of the earlier results for the baryon axial currents, masses and magnetic moments [1,2,3,4,10,11]. In addition, new results are presented for the three flavor case in the symmetry limit, and to first order in symmetry breaking. New results are also presented for the hyperon non-leptonic decay amplitudes. To make the results accessible to a wider audience, we will present detailed comparisons of the large N_c predictions with the experimental data in another paper [12].

The operator analysis in this paper is discussed almost entirely using the quark representation. The connection with the Skyrme model is discussed in Section XIII. The quark representation uses the algebraic structure of the non-relativistic quark model to classify all the baryon operators. It is important to stress, however, that the results of this paper do not assume that the non-relativistic quark model is valid, or that the quarks in the baryon are non-relativistic. A more detailed discussion of the connection between the quark basis and large N_c QCD can be found in refs. [3,10,11]. Finally, we restrict our analysis principally to the ground-state baryons. Excited baryons have been considered in ref. [13].

II. THE QUARK REPRESENTATION

The quark representation of the spin-flavor symmetry of large N_c baryons is based on the non-relativistic quark model picture. However, as emphasized in the introduction, using the quark model realization of the contracted spin-flavor symmetry does not mean that we are treating the quarks in the baryon as non-relativistic. The non-relativistic quark model algebra provides a convenient way of writing the results of a $1/N_c$ calculation in QCD, which are valid even for baryons with massless quarks. We will refer to the quark representation rather than the non-relativistic quark model, to emphasize this distinction.

In the quark representation, one defines a set of quark creation and annihilation operators, $q^\dagger_\alpha$ and q^α, where $\alpha = 1, \ldots, F$ represents the F quark flavors with spin up, and $\alpha = F+1, \ldots, 2F$, the F quark flavors with spin down. The antisymmetry of the $SU(N_c)$ color ϵ-symbol and Fermi statistics implies that the ground-state baryons contain N_c quarks in the completely symmetric representation of spin $\otimes$ flavor (see Fig. 1), so one can omit the color quantum numbers of the quark operators for the spin-flavor analysis and treat them as bosonic objects. Thus, the quark operators satisfy the bosonic commutation relation

$$\left[q^\alpha, q^\dagger_\beta\right] = \delta^\alpha_\beta. \tag{2.1}$$

In this work, we consider the spin-flavor structure of the ground-state baryons for N_c large and finite, and odd. The completely symmetric N_c-quark representation of $SU(2F)$ contains baryons with spin $1/2$, $3/2$, ... , $N_c/2$, which transform as the flavor representations shown in Table I, respectively. For two flavors, the baryon states can be labeled by their spin J and isospin I, $(J, I) = (1/2, 1/2)$, $(3/2, 3/2)$, ..., $(N_c/2, N_c/2)$. For three flavors, the spin-1/2 baryons have the weight diagram shown in Fig. 2, and the spin-3/2 baryons have the weight diagram shown in Fig. 3. Generically, the spin J weight diagram has an edge with $2J+1$ weights, and an edge with $(N_c+2)/2-J$ weights. The multiplicity starts at one for the outermost weights, and increases by one as one moves inward, until one reaches the point at which the weights are triangular. From this point inwards, the multiplicity remains constant. The dimension of the representation is $ab(a+b)/2$, where a and b are the number of weights on the two edges. The weight diagrams of

the spin-1/2 and spin-3/2 baryons reduce to the baryon octet and decuplet for $N_c = 3$. For $F > 2$, the baryon flavor representations grow rapidly with N_c, and are not the same as the flavor representations for $N_c = 3$. This dependence of the flavor representations on N_c leads to subtleties in obtaining results for $N_c = 3$.

Quark operators can be classified according to whether they are zero-body, one-body, ..., or n-body operators. A zero-body operator contains no q or $q^\dagger$. There is a unique zero-body operator, the identity operator $\mathbb{1}$. A one-body operator acts on a single quark. The one-body operators consist of the quark number operator $q^\dagger q$ and the spin-flavor adjoint $q^\dagger \Lambda^A q$, $A = 1, \ldots, (2F)^2 - 1$, where Λ^A is a spin-flavor generator. Two-body operators involve two q's and two $q^\dagger$'s, and act upon two quarks. Two-body operators can be written either as bilinears in the one-body operators or in normal ordered form (e.g. $q^\dagger q^\dagger q q$). Normal ordered two-body operators are "pure" two-body operators, in the sense that they have vanishing matrix elements on single quark states. Similarly, one can consider n-body operators either as polynomials of degree n in the one-body operators, or in normal ordered form with n q's and n $q^\dagger$'s. n-body operators acting on a N_c-quark state typically have matrix elements of order N_c^n because of combinatoric factors associated with inserting the operator in the N_c-quark state.

Since we eventually will be interested in classifying operators according to their spin and flavor representations, it is convenient to decompose the $SU(2F)$ adjoint one-body operator $q^\dagger \Lambda^A q$ into representations of $SU(2) \times SU(F)$,

$$\begin{aligned}
J^i &= q^\dagger \left(J^i \otimes \mathbb{1}\right) q \qquad (1,0),\\
T^a &= q^\dagger \left(\mathbb{1} \otimes T^a\right) q \qquad (0,adj),\\
G^{ia} &= q^\dagger \left(J^i \otimes T^a\right) q \qquad (1,adj),
\end{aligned} \tag{2.2}$$

where J^i are the spin generators, T^a are the flavor generators, and G^{ia} are the spin-flavor generators. The transformation properties of these generators under $SU(2) \times SU(F)$ are given in eq. (2.2), where adj denotes the F^2-1 dimensional adjoint representation of $SU(F)$. Throughout this work, uppercase letters (A, B, ...) denote indices transforming according to the adjoint representation of the $SU(2F)$ spin-flavor group, lowercase letters (a, b, ...) denote indices transforming according to the adjoint representation of the $SU(F)$ flavor group, and (i, j, ...) denote indices transforming according to the vector representation of spin. The matrices J^i and T^a on the right-hand side of eq. (2.2) are in the fundamental representations of $SU(2)$ and $SU(F)$, respectively, and are normalized so that

$$\begin{aligned}
\mathrm{Tr}\ J^i J^j &= \frac{1}{2}\,\delta^{ij},\\
\mathrm{Tr}\ T^a T^b &= \frac{1}{2}\,\delta^{ab}.
\end{aligned} \tag{2.3}$$

The spin-flavor matrices Λ^A normalized to

$$\mathrm{Tr}\ \Lambda^A \Lambda^B = \frac{1}{2}\,\delta^{AB}, \tag{2.4}$$

are $\left(J^i \otimes \mathbb{1}\right)/\sqrt{F}$, $\left(\mathbb{1} \otimes T^a\right)/\sqrt{2}$ and $\sqrt{2}\left(J^i \otimes T^a\right)$, so that the properly normalized $SU(2F)$ operators are $J^i/\sqrt{F}$, $T^a/\sqrt{2}$ and $\sqrt{2}\,G^{ia}$.

III. LARGE N_c POWER COUNTING

The baryons in QCD are color singlet states of N_c quarks. The N_c-dependence of operator matrix elements in baryon states can be obtained using the double line notation of 't Hooft [15]. The N_c counting rules were discussed extensively by Witten [16], and more recently in refs. [10,11]. The basic result can be given very simply using an illustrative example.

Consider the baryon matrix element of a one-quark QCD operator $\mathcal{O}_{QCD} = \bar{q}\Gamma q$, where Γ is a Dirac and flavor matrix.* For example, the operator could be the flavor singlet axial current, with $\Gamma = \gamma_\mu\gamma_5$, or a flavor octet vector current, with $\Gamma = T^a\gamma_\mu$, etc. The baryon matrix element of $\mathcal{O}_{QCD}$ is obtained by inserting the operator on any of the N_c quark lines, as show in Fig. 4a. There are N_c insertions, and each graph is of order one, so that a one-quark QCD operator has a matrix element which is at most of order N_c. The matrix element is not necessarily of order N_c, however, since there may be cancellations among the N_c insertions on the various quark lines. All planar graphs with additional gluon exchanges (Fig. 4b) are of the same order in N_c as Fig. 4a, whereas graphs with additional exchanges of non-planar gluons are suppressed by powers of $1/N_c$ relative to Fig. 4a.

The QCD operator is given by an expansion in $1/N_c$ in terms of operators in the quark representation. At leading order, the QCD operator has an expansion of the form

$$\mathcal{O}_{QCD} = \sum_{n,k} c_k^{(n)} \frac{1}{N_c^{n-1}} \mathcal{O}_k^{(n)}, \tag{3.1}$$

where the sum is over all possible n-body operators $\mathcal{O}_k^{(n)}$, $n = 0, \ldots, N_c$, with the same spin and flavor quantum numbers as $\mathcal{O}_{QCD}$, with coefficients $c_k^{(n)}$ of order unity. Subleading $1/N_c$ corrections to the leading order expression (3.1) can be included by adding $1/N_c$ corrections to the coefficients $c_k^{(n)}$. The complicated QCD dynamics is parametrized by the unknown coefficients $c_k^{(n)}$. A comparison of the form of this expansion with the Feynman diagrams in Fig. 4 is suggestive. The one-body operator can be thought of as arising from the operator insertion graphs depicted in Fig. 4a. The single gluon exchange

*Note that the quark field q in the QCD operator is not the same as the quark operator q of the quark representation.

graphs of Fig. 4b produce two-body operators with an extra factor of $1/N_c$ from the two gauge coupling constants at the gluon vertices, and so on. Non-planar gluon exchange graphs result in $1/N_c$ corrections to the operator coefficients. There are N_c quarks in the baryon, so one can terminate the expansion at N_c-body operators. An n-body operator is typically of order N_c^n, so that all of the terms in eq. (3.1) are of the same order in the $1/N_c$ expansion as the leading term. In the limit $N_c \to \infty$, one obtains an infinite series of operators which are equally important even at leading order in $1/N_c$. Since we cannot evaluate the coefficients $c_k^{(n)}$, the entire $1/N_c$ expansion would be intractable, were it not for a series of operator identities which allows the number of operators to be reduced to a finite set at a given order in $1/N_c$. In this paper, we derive all these operator identities, and classify the independent operators at any given order in $1/N_c$ for some quantities of interest, such as the baryon axial currents, masses, magnetic moments and non-leptonic decay amplitudes.

An important feature of the above N_c counting for the n-body quark operators is that the N_c counting is preserved under commutation. The commutator of an m-body operator with an n-body operator is an $(m+n-1)$-body operator,

$$\left[\mathcal{O}^{(m)}, \mathcal{O}^{(n)}\right] = \mathcal{O}^{(m+n-1)}, \tag{3.2}$$

and $\left(1/N_c^{m-1}\right)\left(1/N_c^{n-1}\right) = \left(1/N_c^{(m+n-1)-1}\right)$. In contrast, the anticommutator of an m-body and n-body operator is typically an $(m+n)$-body operator. The commutator has one less $q^\dagger q$ than the anticommutator, because quark operators acting on different quark lines commute. The commutativity of quark operators acting on different quark lines forces one quark in $\mathcal{O}^{(m)}$ to act on the same quark line as a quark in $\mathcal{O}^{(n)}$ to produce a non-zero commutator, and reduces the $(m+n)$-body operator to an $(m+n-1)$-body operator.

We have given the quark counting rules for a one-quark QCD operator. Similarly, it is easy to see that a m-quark QCD operator is given as an expansion in terms of n-body operators with coefficients of order N_c^{m-n}. It need not be the case that $n \geq m$. For example, in $\Delta I = 1/2$ weak decays, a four-quark (i.e. two-body) QCD operator can produce a one-body quark operator.[†]

IV. QUARK OPERATOR IDENTITIES: CLASSIFICATION

In this section, we classify all independent operator identities among the n-body operators in the quark representation. These identities have an elegant group theoretical structure. Readers not interested in the details can look at the identities for three flavors in Table VIII, and skip to the operator reduction rule at the end of Section VII. The general structure of the identities is that certain n-body operators can be reduced to linear combinations of m-body operators, where $m < n$. Since n-body operators acting on an N_c-quark baryon state are generically of order N_c^n, the coefficient of the m-body operator is typically of order N_c^{n-m}. For example, some three-body operators can be reduced to two-body operators with coefficients of order N_c, one-body operators with coefficients of order N_c^2, and zero-body operators with coefficients of order N_c^3. We will show that the only independent operator identities which are required are those which reduce two-body operators to linear combinations of one-body and/or zero-body operators. All identities for n-body operators with $n > 2$ can be obtained by recursively applying two-body identities. This result leads to a tremendous simplification in the analysis, since there are only a finite number of identities which need to be written explicitly for the two-body case. Explicit expressions for the two-body identities are derived in Section V, and are given in Table VI for an arbitrary number of flavors, and in Tables VII and VIII for two and three flavors, respectively.

A. Zero-Body Operators

There is a unique zero-body operator, the identity operator $\mathbb{1}$, which has matrix elements $N_c^0 = 1$. The identity operator transforms as a singlet under the spin-flavor group $SU(2F)$ and as a singlet under $SU(2) \times SU(F)$. There are no operator identities at this level.

B. One-Body Operators

The one-body operators transform under $SU(2F)$ as the tensor product of a quark and antiquark representation. A quark is in the fundamental representation of $SU(2F)$, and transforms as a tensor with one upper index. The antiquark transforms as an $SU(2F)$ tensor with one lower index. Thus, the one-body operators transform as

$$1-\text{body}: \ \left(\overline{\square} \otimes \square\right) = 1 + adj = 1 + T^\alpha_\beta, \tag{4.1}$$

where T^α_β is a traceless tensor which transforms as the adjoint representation of $SU(2F)$.

The independent one-body operators were listed in Section II; they are the quark number operator $q^\dagger q$ and the spin-flavor operators $q^\dagger \Lambda^A q$, which consist of J^i, T^a, and G^{ia}. The quark number operator $q^\dagger q$ is a singlet under $SU(2F)$ and J^i, T^a and G^{ia} together form the adjoint representation of $SU(2F)$, which agrees with the

[†]A similar result was found in the chiral quark model [17].

analysis of eq. (4.1). These one-body operators transform as $(0,0)$, $(1,0)$, $(0,adj)$ and $(1,adj)$, respectively, under $SU(2)\times SU(F)$.

The only operator identity allowed at this stage is one relating the one-body and zero-body $SU(2F)$ singlets. This identity is trivial,

$$q^\dagger q = N_c\ \mathbb{1}. \tag{4.2}$$

Note that this identity has the general structure stated at the beginning of this section: a one-body operator is written as N_c times a zero-body operator.

C. Two-Body Operators

The non-trivial identities occur among two-body operators. The two-body operators transform as the tensor product of a two-quark and two-antiquark state. Since the quarks in the ground-state baryon representation are in a completely symmetric state (Fig. 1), any two quarks transform according to the two-index symmetric tensor representation of $SU(2F)$, and any two antiquarks are in the complex conjugate representation. Thus, the two-body operators transform as

$$\begin{aligned} 2-\text{body}:\ \left(\overline{\square\square}\otimes\square\square\right) &= 1 + adj + \bar{s}s \\ &= 1 + T^\alpha_\beta + T^{(\alpha_1\alpha_2)}_{(\beta_1\beta_2)}, \end{aligned} \tag{4.3}$$

where $T^{(\alpha_1\alpha_2)}_{(\beta_1\beta_2)}$ is a traceless tensor which is completely symmetric in its upper and lower indices. This tensor representation of $SU(2F)$ will be called the $\bar{s}s$ representation.

It is convenient to write the two-body operators as products of two one-body operators, rather than to write them in normal-ordered form directly. The quark number operator $q^\dagger q$ can be eliminated using the identity eq. (4.2), so we only need to consider bilinears of the $SU(2F)$ adjoint representation $q^\dagger\Lambda^A q$, which consists of J^i, T^a and G^{ia}. Any product of two operators can always be written as the symmetric product (an anticommutator), or the antisymmetric product (a commutator). The commutator can be eliminated using the $SU(2F)$ Lie algebra commutation relations listed in Table II. The anticommutator transforms as the symmetric product of two $SU(2F)$ adjoints,

$$\left(adj\otimes adj\right)_S = 1 + adj + \bar{a}a + \bar{s}s, \tag{4.4}$$

where $\bar{a}a = T^{[\alpha_1\alpha_2]}_{[\beta_1\beta_2]}$ transforms as a traceless tensor which is antisymmetric in its upper and lower indices. The decomposition of the symmetric tensor product of two adjoints for an arbitrary $SU(Q)$ group, and for the special cases $Q=6$ and $Q=4$ are listed in Table III, using the Dynkin notation for the irreducible representations. Each of the representations in $\left(adj\otimes adj\right)_S$ occurs in eq. (4.3) except for the $\bar{a}a$ representation.

The structure of all two-body identities can now be determined. We quote the results here; a detailed derivation of all the identities is presented in Sec. V. The two-body identities can be divided into three different sets:

1. There is a linear combination of two-body operators which is an $SU(2F)$ singlet. This linear combination can be written as a coefficient of order N_c^2 times the zero-body unit operator $\mathbb{1}$. The $SU(2F)$ singlet in $\left(adj\otimes adj\right)_S$ is the Casimir operator, which equals

$$\left\{q^\dagger\Lambda^A q, q^\dagger\Lambda^A q\right\} = N_c\left(N_c+2F\right)\left(1-\frac{1}{2F}\right)\mathbb{1}, \tag{4.5}$$

where the coefficient of the $\mathbb{1}$ operator is the $SU(2F)$ Casimir for the completely symmetric baryon representation Fig. 1.

2. There is a linear combination of two-body operators which transforms as an $SU(2F)$ adjoint. This linear combination can be written as a coefficient of order N_c times the one-body adjoint operator. The $SU(2F)$ adjoint in $\left(adj\otimes adj\right)_S$ is obtained by contraction with the $SU(2F)$ d-symbol d^{ABC}; it equals

$$d^{ABC}\left\{q^\dagger\Lambda^B q, q^\dagger\Lambda^C q\right\} = 2\left(N_c+F\right)\left(1-\frac{1}{F}\right)\ q^\dagger\Lambda^A q\ , \tag{4.6}$$

for the completely symmetric baryon representation. The coefficient on the right-hand side of eq. (4.6) is the ratio of the cubic and quadratic Casimirs of the completely symmetric baryon representation Fig. 1.

3. Comparison of eqs. (4.3) and (4.4) shows that there is no $\bar{a}a$ representation for two-body operators acting on the completely symmetric baryon representation, but there is an $\bar{a}a$ representation in $\left(adj\otimes adj\right)_S$. Thus, the linear combination of bilinears in one-body operators which transforms as an $\bar{a}a$ must vanish for the completely symmetric baryon representation. This set of identities eliminates certain bilinears in $\{J^i, T^a, G^{ia}\}$ from the set of independent two-body operators.

D. Three-Body Operators and Generalization

Three-body operators act on symmetric tensor products of three-quark states, and transform as the representations

$$\begin{aligned} &3-\text{body}:\ \left(\overline{\square\square\square}\otimes\square\square\square\right) \\ &= 1 + T^\alpha_\beta + T^{(\alpha_1\alpha_2)}_{(\beta_1\beta_2)} + T^{(\alpha_1\alpha_2\alpha_3)}_{(\beta_1\beta_2\beta_3)}. \end{aligned} \tag{4.7}$$

The only new tensor occurring at three-body is the traceless tensor $T^{(\alpha_1\alpha_2\alpha_3)}_{(\beta_1\beta_2\beta_3)}$. Any three-body operator can be written as a trilinear in the one-body operators. The one-body quark number operator can be trivially replaced by $N_c\mathbb{1}$, so only the adjoint one-body operators need to be considered. Any trilinear product of adjoint one-body operators which is not completely symmetric can be reduced to two-body operators using the $SU(2F)$ commutation relations given in Table II, so one only needs to consider completely symmetric trilinears in the adjoint one-body operators. The decomposition of $(adj\otimes adj\otimes adj)_S$ is given in Table IV for a general $SU(Q)$ group and for the special cases $Q=6$ and $Q=4$.

First consider operator identities which relate the three-body singlet, adjoint and $\bar{s}s$ representations to zero-body, one-body, and two-body operators times coefficients of order N_c^3, N_c^2, and N_c, respectively. In normal ordered form, it is easy to see that these three-body identities are obtained by contraction of a pair of $q^\dagger$, q indices. Contraction of pairs of quark indices is already described by the two-body $\rightarrow$ one-body and two-body $\rightarrow$ zero-body identities, so judicious application of these identities yields the required three-body identities.

Table IV shows that there are ten irreducible $SU(2F)$ representations in $(adj\otimes adj\otimes adj)_S$ for $F\geq 2$, of which only four are present in eq. (4.7). Thus, in principle, there are six sets of identities which vanish identically for the three-body case. (For $F=2$, there are eight irreducible $SU(2F)$ representations in $(adj\otimes adj\otimes adj)_S$, and thus, in principle, four sets of identities which vanish identically for the three-body case.) It is clear, however, that not all of these identities are really new, since at least some of the three-body identities are simply products of two-body identities in the $\bar{a}a$ representation and one-body operators in the adjoint representation, J^i, T^a, or G^{ia}. The tensor product of $\bar{a}a$ with the adjoint representation is given in Table V. Comparison of Tables IV and V shows that all the representations in $(adj\otimes adj\otimes adj)_S$ which are not present in eq. (4.7) occur in $(\bar{a}a\otimes adj)$. This observation is not sufficient to conclude that all the three-body identities are given in terms of two-body identities, however. The three-body representations in Table IV are contained in the completely symmetric tensor product of three adjoints, $(adj\otimes adj\otimes adj)_S$. The representations in $(\bar{a}a\otimes adj)$ of Table V are in the tensor product $(adj\otimes adj)_S\otimes adj$, since $\bar{a}a$ is contained in $(adj\otimes adj)_S$. To find the representations which reduce to the two-body $\bar{a}a$ identities, one has to impose the additional constraint that the three adjoints in $(\bar{a}a\otimes adj)$ are completely symmetric. Not all the irreducible representations of Table V survive when this constraint is imposed, but all the irreducible representations of Table IV which are not present in eq. (4.7) do survive, as can be checked explicitly. This observation leads to the conclusion that there are no new vanishing three-body identities which are not simply products of the one-body and two-body identities which have already been determined.

This conclusion can be generalized to n-body operators. There are no new vanishing identities for n-body operators, $n\geq 3$, which are not products of the original $\bar{a}a$ two-body identities and one-body operators. The $\bar{a}a$ two-body identities result because any product of two adjoint one-body operators which is antisymmetric in creation or annihilation operators must vanish when it acts on the completely symmetric baryon representation. Similarly, the vanishing n-body identities result from products of n one-body operators which are not completely symmetric in the n creation and n annihilation operators. Any representation of the permutation group which is not completely symmetric in all the quark creation (or annihilation) operators must be antisymmetric in at least one pair. Two one-body operators containing an antisymmetric quark pair vanish by the two-body identities derived earlier, so n-body operators containing an antisymmetric quark pair automatically vanish by the two-body identities, and there are no new identities which vanish for $n\geq 3$.

In summary, we have classified all non-trivial operator identities for $SU(2F)$ quark operators. For n-body quark operators, the only representations of the $SU(2F)$ group which are allowed are $1+T^{\alpha_1}_{\beta_1}+T^{(\alpha_1\alpha_2)}_{(\beta_1\beta_2)}+\ldots+T^{(\alpha_1\alpha_2\ldots\alpha_n)}_{(\beta_1\beta_2\ldots\beta_n)}$. All other representations can be eliminated using the two-body $\bar{a}a$ operator identities. Furthermore, the only "purely" n-body representation is $T^{(\alpha_1\alpha_2\ldots\alpha_n)}_{(\beta_1\beta_2\ldots\beta_n)}$. The non-vanishing two-body operator identities can be used to write n-body operators that transform as $T^{(\alpha_1\alpha_2\ldots\alpha_m)}_{(\beta_1\beta_2\ldots\beta_m)}$ $(m<n)$ as m-body operators times coefficients of order N_c^{n-m}.

V. TWO-BODY QUARK IDENTITIES: DERIVATION

In this section, all of the non-trivial two-body operator identities are derived explicitly for an arbitrary number of light quark flavors. The identities are listed in Table VI. The $SU(2F)$ group theory which is needed for the computation is given in appendix A.

A. Two-Body $\rightarrow$ Zero-Body Identity

The $SU(2F)$ singlet in the symmetric product of two $SU(2F)$ generators is the Casimir operator $\Lambda^A\Lambda^A$, which is a constant for a given irreducible representation. The Casimir for an $SU(Q)$ irreducible representation R (see [18]) is

$$C_2(R)=\frac{1}{2}\left(NQ-\frac{N^2}{Q}+\sum_i r_i^2-\sum_i c_i^2\right), \qquad (5.1)$$

where r_i is the number of boxes in the $\imath^{\rm th}$ row of the Young tableau, c_i is the number of boxes in the $\imath^{\rm th}$ column of the Young tableau, and $N = \sum_i r_i = \sum_i c_i$ is the total number of boxes. The Casimir for the completely symmetric $SU(2F)$ baryon representation with a single row of N_c boxes (Fig. 1) is

$$C_2 = \frac{1}{2} N_c \left(N_c + 2F\right) \left(1 - \frac{1}{2F}\right), \tag{5.2}$$

so the Casimir identity‡ is

$$\left\{q^\dagger \Lambda^A q, \quad q^\dagger \Lambda^A q\right\} = N_c \left(N_c + 2F\right) \left(1 - \frac{1}{2F}\right) \mathbb{1}. \tag{5.3}$$

The Casimir operator $\Lambda^A \Lambda^A$ equals $J^2/F + T^2/2 + 2\, G^2$ using the properly normalized $SU(2F)$ generators. Combining this relation with eq. (5.3) gives the first identity in Table VI. Note that the coefficient of the zero-body operator is of order N_c^2, as expected.

B. Two-Body → One-body Identity

The linear combination of two-body operators which transforms as an $SU(2F)$ adjoint reduces to the adjoint one-body operator

$$d^{ABC} \left\{q^\dagger \Lambda^A q, q^\dagger \Lambda^B q\right\} = D(R)\; q^\dagger \Lambda^C q, \tag{5.4}$$

where $D(R)$ is a constant which must be determined for the completely symmetric baryon representation. The two-body operator in eq. (5.4) can be written as

$$\begin{aligned} &2\, d^{ABC} \sum_{r,s=1}^{N_c} q^\dagger_{r\alpha} \left(\Lambda^A\right)^\alpha_\beta q^\beta_r \; q^\dagger_{s\gamma} \left(\Lambda^B\right)^\gamma_\zeta q^\zeta_s \\ &= 2\, d^{ABC} \; \left(\Lambda^A\right)^\alpha_\beta \left(\Lambda^B\right)^\gamma_\zeta \sum_{r,s=1}^{N_c} q^\dagger_{r\alpha} q^\beta_r q^\dagger_{s\gamma} q^\zeta_s, \end{aligned}$$

where q_r denotes the quark annihilation operator acting on the $r^{\rm th}$ quark in the baryon, and the sums on r and s run over the N_c quarks in the baryon. Using the $SU(2F)$ identity

$$\begin{aligned} d^{ABC} \left(\Lambda^A\right)^\alpha_\beta \left(\Lambda^B\right)^\gamma_\zeta = &-\frac{1}{2F} \left[\delta^\alpha_\beta \left(\Lambda^C\right)^\gamma_\zeta + \left(\Lambda^C\right)^\alpha_\beta \delta^\gamma_\zeta\right] \\ &+\frac{1}{2} \left[\delta^\alpha_\zeta \left(\Lambda^C\right)^\gamma_\beta + \left(\Lambda^C\right)^\alpha_\zeta \delta^\gamma_\beta\right], \end{aligned} \tag{5.5}$$

the two-body operator can be rewritten as

$$\begin{aligned} &-\frac{2N_c}{F} q^\dagger \Lambda^C q + \sum_{r,s} \left(\Lambda^C\right)^\gamma_\beta q^\dagger_{r\alpha} q^\beta_r q^\dagger_{s\gamma} q^\alpha_s \\ &+\sum_{r,s} \left(\Lambda^C\right)^\alpha_\zeta q^\dagger_{r\alpha} q^\beta_r q^\dagger_{s\beta} q^\zeta_s \; . \end{aligned} \tag{5.6}$$

The first term is in the form required by eq. (5.4), but the last two terms need further simplification. The summation over r and s for these terms can be divided into a sum over $r = s$, and over $r \neq s$. The terms with $r = s$ in eq. (5.6) are

$$\sum_r \left(\Lambda^C\right)^\gamma_\beta q^\dagger_{r\alpha} q^\beta_r q^\dagger_{r\gamma} q^\alpha_r + \sum_r \left(\Lambda^C\right)^\alpha_\zeta q^\dagger_{r\alpha} q^\beta_r q^\dagger_{r\beta} q^\zeta_r. \tag{5.7}$$

The normal ordered version of this operator vanishes, since each quark in the baryon is only singly occupied (any two annihilation operators $q^\alpha_r q^\beta_r$ acting on the same quark line must vanish (even if $\alpha \neq \beta$)). Normal ordering eq. (5.7) using the quark commutator eq. (2.1) yields

$$\begin{aligned} &\sum_r \left(\Lambda^C\right)^\gamma_\beta q^\dagger_{r\alpha} q^\alpha_r \delta^\beta_\gamma + \sum_r \left(\Lambda^C\right)^\alpha_\zeta q^\dagger_{r\alpha} q^\zeta_r \delta^\beta_\beta \\ &= 2F \sum_r q^\dagger_{r\alpha} \left(\Lambda^C\right)^\alpha_\zeta q^\zeta_r = 2F \; q^\dagger \Lambda^C q \end{aligned} \tag{5.8}$$

where the first term vanishes because Λ^C is traceless and $\delta^\beta_\beta = 2F$ for the second term. Finally, the contribution of the last two terms in eq. (5.6) with $r \neq s$ must be evaluated. Since q_r and $q^\dagger_s$ act on different quark lines, they can be treated as commuting operators. Thus the two sums in eq. (5.6) with $r \neq s$ are equal to each other (as can be seen by exchanging the dummy indices r and s), and their sum is equal to

$$2 \sum_{r \neq s} \left(\Lambda^C\right)^\gamma_\beta q^\dagger_{r\alpha} q^\dagger_{s\gamma} q^\beta_r q^\alpha_s. \tag{5.9}$$

This operator acts on the (r, s) quark pair in the initial baryon. The initial baryon is completely symmetric in flavor, so one can exchange the flavor labels α and β on the initial quark pair. This gives the equivalent operator

$$\begin{aligned} &2 \sum_{r \neq s} \left(\Lambda^C\right)^\gamma_\beta q^\dagger_{r\alpha} q^\dagger_{s\gamma} q^\alpha_r q^\beta_s = 2 \sum_{r \neq s} q^\dagger_{s\gamma} \left(\Lambda^C\right)^\gamma_\beta q^\beta_s q^\dagger_{r\alpha} q^\alpha_r \\ &= 2 \left(N_c - 1\right) q^\dagger \Lambda^C q, \end{aligned} \tag{5.10}$$

since there are $(N_c - 1)$ values of $r \neq s$ for a given value of s, and $q^\dagger_r q_r$ is unity since each quark line is singly occupied. Combining eqs. (5.6)–(5.10) with eq. (5.4) gives the final form of the identity

$$d^{ABC} \left\{q^\dagger \Lambda^A q, q^\dagger \Lambda^B q\right\} = 2 \left(N_c + F\right) \left(1 - \frac{1}{F}\right) q^\dagger \Lambda^C q. \tag{5.11}$$

The identity eq. (5.11) for the $SU(2F)$ adjoint two-body operator can be decomposed under $SU(2) \times SU(F)$

‡The Casimir operator in the completely symmetric baryon representation can also be computed directly using the quark operators and the Fierz identity eq. (A12).

into the three representations $(1,0)$, $(0,adj)$ and $(1,adj)$. The identities for these three $SU(2)\times SU(F)$ representations can be obtained by substituting in the properly normalized $SU(2F)$ generators $J^i/\sqrt{F}$, $T^a/\sqrt{2}$, and $\sqrt{2}\,G^{ia}$ into eq. (5.11), and using the decomposition of the $SU(2F)$ d-symbol under $SU(2)\times SU(F)$ given in eq. (B4). Eq. (5.11) then yields the three identities in the second block of Table VI.

C. Vanishing Two-Body Operators

The final set of identities is obtained by combining the two adjoint one-body operators into the $\bar{a}a$ representation, and setting it equal to zero. The $\bar{a}a$ representation can be obtained by using the $SU(2F)$ projection operators discussed in appendix A. The decomposition of the $\bar{a}a$ representation into irreducible $SU(2)\times SU(F)$ representations is given in eq. (B1). Another method for obtaining the vanishing two-body identities is to simplify anticommutators of J^i, T^a and G^{ia} using the same techniques applied in the derivation of eq. (5.11). One then finds linear combinations of the anticommutators which vanish. The $\bar{a}a$ identities are contained in the third block of Table VI.

The simplest method for obtaining the vanishing two-body identities uses a trick. The baryon $SU(2)\times SU(F)$ representations in the completely symmetric $SU(2F)$ representation have identical Young tableaux for the spin and flavor subgroups (see Table I). Eq. (5.1) then implies that the $SU(F)$ Casimir operator T^aT^a of an arbitrary baryon flavor representation is simply related to its $SU(2)$ Casimir J^iJ^i,

$$T^2 = J^2 + \frac{1}{4F} N_c\left(N_c+2F\right)\left(F-2\right). \tag{5.12}$$

This relation, which transforms as an $SU(2)\times SU(F)$ singlet, is a linear combination of the Casimir identity eq. (5.3) and the $(0,0)$ element of the $\bar{a}a$ representation of $SU(2F)$. The singlet of the $\bar{a}a$ representation is obtained by finding the linear combination of eq. (5.12) and the Casimir identity which orthogonal to the Casimir identity. This linear combination,

$$\begin{aligned} &4F\left(2-F\right)\ \left\{G^{ia},G^{ia}\right\} + 3F^2\ \left\{T^a,T^a\right\} \\ &\quad +4\left(1-F^2\right)\ \left\{J^i,J^i\right\} = 0, \end{aligned} \tag{5.13}$$

is the first identity in the third block of Table V. All the other elements of the $SU(2F)$ $\bar{a}a$ irreducible representation can be obtained by applying raising and lowering operators to the $(0,0)$ element, eq. (5.13). In our case, all other elements of the $\bar{a}a$ representation can be obtained by commuting identity eq. (5.13) with the generators J^i, T^a and G^{ia}. Since J^i and T^a are spin and flavor generators, they do not produce identities which are in new $SU(2)\times SU(F)$ representations. Thus, only commutators of G^{ia} with eq. (5.13) need to be evaluated to obtain the other $SU(2)\times SU(F)$ identities in the $SU(2F)$ $\bar{a}a$ representation. Applying successive commutators, and projecting onto definite $SU(2)\times SU(F)$ channels gives the remaining identities in Table VI. For ease of notation, we have not explicitly written some of the spin and flavor projectors in Table VI, but have simply indicated that both sides of a given equation are to be projected into the relevant channel. For example, $\left\{G^{ia},G^{ja}\right\}(J=2)$ indicates that only the spin-two piece is to be retained, and is a shorthand notation for $\left\{G^{ia},G^{ja}\right\} - (1/3)\delta^{ij}\left\{G^{ka},G^{ka}\right\}$.

VI. THE OPERATOR IDENTITIES FOR TWO AND THREE FLAVORS

The group theoretic structure of the independent quark operators and the complete set of non-trivial quark operator identities were derived in the previous two sections for an arbitrary number of flavors. Specialization of these results to two and three flavors is useful for application of this formalism to QCD. There is considerable simplification in the results for two flavors, since many of the $SU(F)$ representations vanish for $F=2$. There is also some simplification for three flavors.

A. Two Flavors

For two light quark flavors, the zero—three-body operators transform according to the $SU(4)$ irreducible representations

$$\begin{aligned} 0-\text{body}:&\ \ (0\times 0) = 1, \\ 1-\text{body}:&\ \ \left(4\times\overline{4}\right) = 1+15, \\ 2-\text{body}:&\ \ \left(\overline{10}\otimes 10\right) = 1+15+84, \\ 3-\text{body}:&\ \ \left(\overline{20}\otimes 20\right) = 1+15+84+300. \end{aligned} \tag{6.1}$$

The symmetric product of two adjoints (see Table III) transforms as

$$\left(15\otimes 15\right)_S = 1+15+20+84, \tag{6.2}$$

so the vanishing two-body identities transform in the 20-dimensional representation of $SU(4)$. The $SU(4)$ singlet, adjoint and $\bar{a}a$ two-body identities are listed in Table VII. The $SU(2)\times SU(2)$ decompositions of the 15, 20 and 84 are given in appendix B.

B. Three Flavors

For three light quark flavors, the zero—three-body operators transform according to the $SU(6)$ irreducible representations

$$
\begin{aligned}
&0-\text{body}: \ (0\times 0)=1,\\
&1-\text{body}: \ \left(6\times \overline{6}\right)=1+35,\\
&2-\text{body}: \ \left(\overline{21}\otimes 21\right)=1+35+405,\\
&3-\text{body}: \ \left(\overline{56}\otimes 56\right)=1+35+405+2695.
\end{aligned}
\tag{6.3}
$$

The symmetric product of two adjoints (see Table III) transforms as

$$
\left(35\otimes 35\right)_S = 1+35+189+405, \tag{6.4}
$$

so the vanishing two-body identities transform in the 189-dimensional representation of $SU(6)$. The $SU(6)$ singlet, adjoint and $\bar{a}a$ two-body identities are listed in Table VIII. The $SU(2)\times SU(3)$ decompositions of the 35, 189 and 405 are given in appendix B. The $\bar{a}s+\bar{s}a$ and $\bar{s}s$ representations of $SU(3)$ are the $10+\overline{10}$ and 27, respectively. The $\bar{a}a$ representation doesn't exist for the $SU(3)$ flavor group.

VII. OPERATOR ANALYSIS FOR THE $1/N_c$ EXPANSION

We now analyze the spin-flavor structure of baryons in large-N_c QCD for N_c finite and odd. Any QCD operator which transforms as an irreducible representation of $SU(2)\times SU(F)$ can be written as an expansion in n-body quark operators, $n=0,\dots,N_c$, which transform according to the same irreducible representation (see eq. (3.1)). The quark operator identities can be used to construct a linearly independent and complete operator basis of n-body operators with the correct transformation properties. The operator basis for any $SU(2)\times SU(F)$ representation contains a finite number of operators. The $1/N_c$ expansion for any QCD operator can be simplified by retaining only those operators in the operator basis which contribute at a given order in $1/N_c$. In this section, the generic structure of the $1/N_c$ expansion for the ground-state baryons is discussed. Sections IX through XII derive $1/N_c$ expansions for certain static properties of baryons. The analysis for two flavors is straightforward and reproduces earlier results. The analysis for three (or more) flavors is much more subtle and leads to many new results, as well as reproducing some old results.

In the $N_c\to\infty$ limit, it has been shown that the baryon states form degenerate irreducible representations of the $SU(2F)$ spin-flavor algebra generated by spin, flavor, and the space components of the axial flavor currents [1,3]. Matrix elements of the axial currents within a given irreducible baryon representation are of order N_c, whereas matrix elements of the axial currents between different irreducible representations are at most of order $\sqrt{N_c}$. The mass of the degenerate baryon multiplet is of order N_c. The degeneracy of the baryon spectrum is broken by $1/N_c$ corrections, and it has been shown that the $1/N_c$ correction to the baryon masses is proportional to J^2 (in the flavor symmetry limit) [2]. This baryon mass spectrum is depicted in Fig. 5. The degenerate baryon $SU(2F)$ multiplet splits into a tower of states with spin $1/2,\dots,N_c/2$. Mass splittings between baryon states at the bottom of the tower (J of order one) are of order $1/N_c$, whereas mass splittings between baryon states at the top of the tower (J of order N_c) are of order one. The mass difference between the baryon states at the bottom and top of the towers is of order N_c, and is of the same order in N_c as the average mass of the baryon multiplet. The $1/N_c$ correction to the baryon masses is of order N_c near the top of the tower, and is not a small perturbation. However, it is small near the bottom of the tower, where the baryons have spins which are of order one. The $1/N_c$ expansion is therefore valid for the lowest spin states in the $SU(2F)$ baryon representation. Thus, our analysis considers baryons in the limit $N_c\to\infty$ with J fixed.

The $1/N_c$ expansion of a QCD operator is in terms of a basis of n-body quark operators, where $n=0,\dots,N_c$. A generic n-body operator can be written as a polynomial of homogeneous degree n in the one-body operators J^i, T^a and G^{ia},

$$
\mathcal{O}^{(n)}=\sum_{\ell,m}\left(J^i\right)^\ell\left(T^a\right)^m\left(G^{ia}\right)^{n-\ell-m}, \tag{7.1}
$$

so the expansion of a QCD one-body operator has the form

$$
\mathcal{O}_{QCD}=\sum_{\ell,m,n}c^{(n)}\frac{1}{N_c^{n-1}}\left(J^i\right)^\ell\left(T^a\right)^m\left(G^{ia}\right)^{n-\ell-m}, \tag{7.2}
$$

where summation over different n-body operators $\mathcal{O}_k^{(n)}$ is implied. The generalization of eq. (7.2) to an m-body QCD operator is clear from the discussion in Section III. An important feature of the operator expansion (7.2) can now be explained. We argued in Sec. III that the matrix elements of one-body operators are typically of order N_c, though they can be smaller if there are cancellations between insertions of the operator on the various quark lines. There is an important example of such a cancellation: the baryon states with a valid $1/N_c$ expansion are restricted to those states for which the matrix element of the one-body operator J is of order one, not of order N_c. As a result, every factor of J on the right hand side of eq. (7.2) comes with a $1/N_c$ suppression.

The quark operator identities can be used to eliminate redundant operators from the expansion eq. (7.2). A complete and independent operator basis can be constructed by recursively applying the two-body quark operator identities of Secs. IV and V. The anticommutator of two one-body operators occurs in the irreducible $SU(2)\times SU(F)$ representations given in the second column of Table IX. For example, the anticommutator of J^i with J^j is a flavor singlet which transforms in the symmetric tensor product of two spin ones, i.e. it is either a $(0,0)$ or a $(2,0)$. A similar analysis yields all the other entries in the second column of Table IX. Some of these

operators can be eliminated using the operator identities listed in Table VI. There are a total of 15 identities which can be used to eliminate 15 operator products in Table IX, leaving only the representations listed in the third column of the table. There is a simple structure to the operator products which remain. Consider the operator products $\{T^a, T^b\}$, $\{T^a, G^{ib}\}$, and $\{G^{ia}, G^{jb}\}$ which each have two adjoint indices. These indices can be contracted using δ^{ab} to give a flavor singlet, or with d^{abc} or f^{abc} to give flavor adjoints. All these contractions are eliminated by the identities. The spin indices in $\{G^{ia}, G^{jb}\}$ can be contracted with δ^{ij} to give spin zero, or with ϵ^{ijk} to give spin one. These contractions are also eliminated using the identities. In addition, the $(1, \bar{a}a)$ in $\{T^a, G^{ib}\}$ and $(2, \bar{a}a)$ in $\{G^{ia}, G^{jb}\}$ can be removed. All other products (including all operator products involving J) remain. To summarize, the reduction of operator products is given by the

Operator Reduction Rule: All operator products in which two flavor indices are contracted using δ^{ab}, d^{abc} or f^{abc}, or two spin indices on G's are contracted using δ^{ij} or ϵ^{ijk} can be eliminated.[§] In addition, the $(1, \bar{a}a)$ in $\{T^a, G^{ib}\}$ and the $(2, \bar{a}a)$ in $\{G^{ia}, G^{jb}\}$ can be eliminated.

The last two exceptional cases are not important for examples of physical interest, since there is no $\bar{a}a$ representation for $F = 2$ or 3. Note that it is possible to choose a different set of independent operators using the two-body identities. The operator reduction rule we have chosen is appealing because it has a nice physical interpretation, which is discussed in Section IX.

The two-flavor case is special, since the d-symbol vanishes for $SU(2)$, so there are some additional simplifications. There is a symmetry between spin and isospin for the two-flavor case. The operator reduction rule for two flavors becomes: All operators in which two spin or isospin indices are contracted with a δ or ϵ-symbol can be eliminated, with the exception of J^2. Note that the inclusion of J^2, but not I^2, in the set of independent operators does not break the symmetry between spin and isospin, because of the identity $I^2 = J^2$.

In Sections IX through XII, the operator reduction rule is used to construct $1/N_c$ expansions for various static properties of baryons. These include baryon axial vector currents, masses, magnetic moments and hyperon nonleptonic decay amplitudes.

VIII. OPERATOR ANALYSIS FOR COMPLETELY BROKEN SU(3) SYMMETRY

The $1/N_c$ expansion also provides information about the spin-flavor structure of baryons to all orders in $SU(3)$ symmetry breaking if the operator analysis is performed for completely broken $SU(3)$ flavor symmetry. In this section, the quark operator identities and the classification of independent n-body operators is analyzed for $SU(2) \times U(1)$ flavor symmetry. For this analysis, it is necessary to decompose the one-body operators J^i, T^a and G^{ia} which transform under $SU(3)$ flavor symmetry into operators with definite isospin and strangeness. We define new one-body operators

$$\begin{aligned}
&I^a = T^a \quad (a = 1, 2, 3), \\
&G^{ia} = G^{ia} \quad (a = 1, 2, 3), \\
&t^\alpha = s^\dagger q^\alpha \quad , \\
&Y^{i\alpha} = s^\dagger J^i q^\alpha \quad , \\
&N_s = s^\dagger s, \\
&J^i_s = s^\dagger J^i s,
\end{aligned} \tag{8.1}$$

where I^a and G^{ia} are isospin one operators, and t^α and $Y^{i\alpha}$ are isospin-1/2 operators. The spin and strangeness quantum numbers of these operators are obvious from the above definitions. The independent one-body operators for completely broken $SU(3)$ symmetry are J^i, I^a, t^α, $t^\dagger_\alpha$, N_s, G^{ia}, $Y^{i\alpha}$, $Y^{\dagger i}_\alpha$, and J^i_s. This set of operators replaces the $SU(3)$ one-body operators J^i, T^a and G^{ia}. Note that t^α and $Y^{i\alpha}$, $\alpha = 1, 2$, correspond to T^a and G^{ia} for $a = 4 - i5, 6 - i7$, respectively. The strange quark number operator N_s and the strange quark spin operator J^i_s originate from T^8 and G^{i8},

$$\begin{aligned}
T^8 &= \frac{1}{2\sqrt{3}} \left(N_c - 3N_s\right), \\
G^{i8} &= \frac{1}{2\sqrt{3}} \left(J^i - 3J^i_s\right).
\end{aligned} \tag{8.2}$$

The operator Lie algebra of Table II is unaffected by $SU(3)$ breaking, so the commutation relations of the one-body operators for $SU(2) \times U(1)$ flavor symmetry can be obtained directly from the $SU(3)$ flavor commutation relations using the above identifications. The commutation relations for $SU(2) \times U(1)$ flavor symmetry are listed in Table X. Note that the spin-flavor algebra contains an $SU(4)$ spin-flavor subgroup with generators J^i_{ud}, I^a and G^{ia}, where J^i_{ud}, the spin operator for the u and d quarks, is linearly related to the spin operators J^i and J^i_s,

$$J^i_{ud} = u^\dagger J^i u + d^\dagger J^i d = J^i - J^i_s. \tag{8.3}$$

The commutation relations in Table X are written in terms of J^i_{ud} and J^i_s, rather than J^i and J^i_s so that the $SU(4)$ spin-flavor symmetry is manifest. The full spin-flavor symmetry of the algebra is $SU(4) \times SU(2) \times U(1)$, where the $SU(2)$ factor is strange quark spin and the $U(1)$ factor is the number of strange quarks.

[§]Operators such as $f^{acg} d^{bch} \left\{T^g, G^{ih}\right\}$ (which contains $i(\bar{s}a - \bar{a}s)$) are different from $\left\{T^a, G^{ib}\right\}$ (which contains $\bar{s}a + \bar{a}s$), and are not removed, since the two indices on $\left\{T^g, G^{ih}\right\}$ are not contracted using a f or d-symbol. Many combinations in which two adjoint indices g and h are contracted with $f^{acg} d^{bch}$, $f^{acg} f^{bch}$, or $d^{acg} d^{bch}$ can be eliminated using eqs. (A21)–(A24).

The two-body operator identities for $SU(4) \times SU(2) \times U(1)$ spin-flavor symmetry can be obtained by decomposing the $SU(6)$ identities in Table VIII into irreducible representations with definite spin J, isospin I, and strange quark number S**. The resulting identities are given in Tables XI and XII. The identities are denoted by their $SU(2) \times SU(2) \times U(1)$ quantum numbers $(J,I)_S$. Table XI contains the $S=0$ identities and Table XII contains the $S=1$ and $S=2$ identities. (The $S=-1$ and $S=-2$ identities are the conjugates of the identities in Table XII.) The identities are written most easily in terms of the spin operators J^i_{ud} and J^i_s.

The anticommutators of two one-body operators occur in the irreducible $SU(2) \times SU(2) \times U(1)$ representations given in the second column of Table XIII. The tables of identities contain a total of 33 operator identities which can be used to eliminate 33 different representations appearing in the second column of Table XIII. The operator products which remain appear in the third column of the table. There are several simplifications which occur as a result of operator reduction. All $YY^\dagger$, $Yt^\dagger$, $tY^\dagger$ and $tt^\dagger$ anticommutators can be eliminated using the operator identities. This implies that independent n-body operators with $\Delta S = 1$ ($\Delta S = -1$) contain only one factor of t or Y ($t^\dagger$ or $Y^\dagger$). It also implies that $\Delta S = 0$ operators can be simplified so that they do not contain t, $t^\dagger$, Y or $Y^\dagger$. All operator combinations in which two isovector indices are contracted with δ^{ab} or ϵ^{abc}, or in which an isovector and isospinor index are contracted with $\left(\frac{\tau^a}{2}\right)^\alpha_\beta$, can be eliminated. In addition, the product of two G's or two Y's or a G and Y in which the spin indices are contracted with a δ^{ij} or ϵ^{ijk} can be eliminated. A few other operators also can be eliminated. To summarize, the reduction of operator products for the spin $\otimes$ flavor group $SU(2) \times SU(2) \times U(1)$ obeys the

Operator Reduction Rule II:

1. All products of the form $t^\alpha t^\dagger_\beta$, $t^\alpha Y^{\dagger i}_\beta$, $Y^{i\alpha} t^\dagger_\beta$ and $Y^{i\alpha} Y^{\dagger j}_\beta$ can be eliminated.

2. All operator products in which two isovector indices are contracted using δ^{ab} or ϵ^{abc}, or an isovector and an isospinor index are contracted with $\left(\frac{\tau^a}{2}\right)^\alpha_\beta$, can be eliminated.

3. All operator products $G^{ia}G^{jb}$, $Y^{i\alpha}G^{jb}$, and $Y^{i\alpha}Y^{j\beta}$ (and their conjugates) in which two spin indices are contracted using δ^{ij} or ϵ^{ijk} can be eliminated.

4. The operators $\{J^i_s, J^i_s\}$, $\{G^{ia}, J^i_{ud}\}$, $\{Y^{i\alpha}, J^i_s\}$ and $i\epsilon^{ijk}\{Y^{i\alpha}, J^j_s\}$ (and their conjugates) can be eliminated.

.

Note that there are operators with two isospinor indices contracted with $\epsilon_{\alpha\beta}$ which can *not* be eliminated, namely $\epsilon_{\alpha\beta}\{t^\alpha, t^\beta\}$ and $i\epsilon_{\alpha\beta}\{t^\alpha, Y^{i\beta}\}$.

Again, it is possible to choose a different set of independent operators using the two-body identities. The operator reduction rule chosen here is one of the simplest.

IX. AXIAL CURRENTS AND MESON COUPLINGS

The results of the previous sections will be used to obtain $1/N_c$ expansions for the baryon axial currents and meson couplings, masses, magnetic moments and hyperon non-leptonic decay amplitudes. The axial currents and meson couplings are considered in this section. We begin by deriving the $1/N_c$ expansion for the baryon axial vector current in the $SU(F)$ flavor symmetry limit. Expansions for the axial currents to first order in $SU(3)$ breaking and for $SU(2) \times U(1)$ flavor symmetry are derived in subsequent subsections.

Only the space components of the axial current have non-zero matrix elements at zero recoil, so the axial vector current A^{ia} transforms as $(1, adj)$ under $SU(2) \times SU(F)$. The $1/N_c$ expansion for A^{ia} derived in this section can be applied to any other baryon operator which transforms as a $(1, adj)$, such as the magnetic moment operator.

The n-body quark operators in the $1/N_c$ expansion of A^{ia} reduce to operators containing only a single factor of G^{ia} or T^a by the operator reduction rule, which eliminates all operator products with contracted flavor indices. The only one-body operator is G^{ia}. There are two allowed two-body operators, which can be written as

$$\mathcal{O}^{ia}_2 = \epsilon^{ijk}\left\{J^j, G^{ka}\right\} = i\left[J^2, G^{ia}\right], \tag{9.1}$$

$$\mathcal{D}^{ia}_2 = J^i T^a. \tag{9.2}$$

There are two three-body operators, which can be written as

$$\mathcal{O}^{ia}_3 = \left\{J^2, G^{ia}\right\} - \frac{1}{2}\left\{J^i, \left\{J^j, G^{ja}\right\}\right\}, \tag{9.3}$$

$$\mathcal{D}^{ia}_3 = \left\{J^i, \left\{J^j, G^{ja}\right\}\right\}. \tag{9.4}$$

All remaining n-body operators can be obtained recursively by applying anticommutators of J^2 to the above operators. For $n \geq 2$, the independent $(n+2)$-body operators are given by

$$\mathcal{O}^{ia}_{n+2} = \left\{J^2, \mathcal{O}^{ia}_n\right\}, \tag{9.5}$$

$$\mathcal{D}^{ia}_{n+2} = \left\{J^2, \mathcal{D}^{ia}_n\right\}. \tag{9.6}$$

The set of operators G^{ia}, $\mathcal{O}^{ia}_n$ and $\mathcal{D}^{ia}_n$, $n = 2, 3, \ldots, N_c$, form a complete set of linearly independent spin-one adjoints. The operators $\mathcal{D}^{ia}_n$ are diagonal operators, in the

**Note that S is defined as strange quark number N_s, not strangeness. Since the strangeness of an s-quark is -1, S is the negative of strangeness.

sense that they have non-zero matrix elements only between states with the same spin. The operators $\mathcal{O}^{ia}_n$ are purely off-diagonal, in the sense that they only connect states with different spin. The operator G^{ia} is neither diagonal or off-diagonal. The operators G^{ia}, $\mathcal{D}^{ia}_n$ and $\mathcal{O}^{ia}_{2m+1}$, $m = 1, 2, \ldots$, are odd under time reversal, whereas the operators $\mathcal{O}^{ia}_{2m}$, $m = 1, 2, \ldots$ are even under time reversal. Since A^{ia} is T-odd, the operators $\mathcal{O}^{ia}_{2m}$ do not occur in the $1/N_c$ expansion of the axial vector current.

The $1/N_c$ expansion for A^{ia} is

$$A^{ia} = a_1 G^{ia} + \sum_{n=2,3}^{N_c} b_n \frac{1}{N_c^{n-1}} \mathcal{D}^{ia}_n + \sum_{n=3,5}^{N_c} c_n \frac{1}{N_c^{n-1}} \mathcal{O}^{ia}_n, \tag{9.7}$$

where the coefficients a_1, b_n and c_n have expansions in powers of $1/N_c$ and are order unity at leading order in the $1/N_c$ expansion.

The expansion for A^{ia} eq. (9.7) has a simple physical interpretation in terms of quark line diagrams (see Fig. 4). An insertion of the axial current operator on a quark line r gives $J^i_r T^a_r$ acting on that quark line. (The subscript r implies that the operator acts only on the $r^{\rm th}$ quark.) Summation over all quark lines yields the operator

$$\sum_r J^i_r T^a_r = G^{ia}.$$

Spin-independent gluon exchange renormalizes the operator G^{ia}. Spin-dependent gluon exchange produces additional factors of J acting on different quark lines. The most general operator structure is a flavor matrix T^a_r on some quark line r, and a product of J's on one or more different quark lines $s_1, \ldots s_\ell$, summed over all possible choices for $r, s_1, \ldots, s_\ell$. Products of J_r on a single quark line can be reduced to at most one factor of J_r since

$$J^i_r J^j_r = \frac{1}{4}\delta^{ij} + i\frac{1}{2}\epsilon^{ijk} J^k_r$$

for the spin-1/2 operator J^i_r. If r is not equal to any of the s_k's, the operator produced after summing over all possible quark combinations is

$$\sum_{r,s_1,\ldots,s_\ell} T^a_r J_{s_1} \ldots J_{s_\ell} = T^a J \ldots J,$$

where the indices on the J's are combined to form a spin-one operator. If r is equal to one of the s_k's, the operator produced after summing over all possible quark combinations is

$$\sum_{r,s_1,\ldots,s_\ell} T^a_r J_r J_{s_1} \ldots J_{s_\ell} = G^{ia} J \ldots J,$$

where the indices on G^{ia} and the J's are combined to form a spin-one operator. The operator expansion eq. (9.7) has this form. The above diagrammatic argument is similar to the one given in ref. [11].

The expansion for A^{ia} eq. (9.7) can be reduced to a finite operator series using the fact that each factor of J comes with a $1/N_c$ suppression. The truncation of the operator series is different for the two flavor and three flavor cases.

A. Meson Couplings in the Flavor Symmetry Limit

1. Two Flavors and the $\mathbf{I} = \mathbf{J}$ *Rule*

In the two-flavor case, the isospin I is of order one for baryons with spin J of order one, since $I = J$. Thus every factor of I and J in eq. (9.7) brings a $1/N_c$ suppression. The matrix elements of the operator G^{ia} are of order N_c, as are the matrix elements of $\mathcal{O}^{ia}_m$ and $\mathcal{D}^{ia}_m$, m odd. The matrix elements of $\mathcal{D}^{ia}_2 = J^i I^a$, and $\mathcal{D}^{ia}_m$, m even, are of order one. Thus, the operator expansion eq. (9.7) can be truncated after the one-body operator G^{ia},

$$A^{ia} = a_1\ G^{ia} \left[1 + \mathcal{O}\left(\frac{1}{N_c^2}\right)\right]. \tag{9.8}$$

Eq. (9.7) implies that there are no $1/N_c$ corrections to ratios of pion-baryon couplings for two quark flavors [1]. It also implies that there are no $1/N_c$ corrections to ratios of pion-baryon couplings amongst baryons in a given strangeness sector for three quark flavors [3].

In addition, one can prove that pion couplings to baryons are purely p-wave in the large N_c limit, which is an example of the $I = J$ rule of Mattis and collaborators [8]. For arbitrary N_c, the baryons form a tower of states with J ranging from 1/2 to $N_c/2$. Neglecting baryon recoil (since the baryon mass is of order N_c), the only pseudoscalar meson coupling between the spin-1/2 and spin-3/2 states is a p-wave ($\ell = 1$) coupling. However, higher spin baryons can have couplings in other angular momentum channels, such as $\ell = 3, 5, \ldots$, where ℓ must be odd by parity. A meson coupling in the $\ell^{\rm th}$ partial wave is given by an operator $A^{(i_1,\ldots,i_\ell)a}$ which is completely symmetric and traceless in the ℓ spin indices. The operator reduction rule implies that at leading order in N_c, this operator is proportional to $G^{i_1 a} J^{i_2} J^{i_3} \ldots J^{i_\ell}/N_c^{\ell-1}$, completely symmetrized in the spin indices, with a correction of relative order $1/N_c^2$. Since matrix elements of G^{ia} are of order N_c, and matrix elements of J^i are of order one, we conclude that the $\ell = 1$ coupling is of order N_c (which is just eq. (9.8)), the $\ell = 3$ coupling is of order $1/N_c$, the $\ell = 5$ coupling is of order $1/N_c^3$, etc. In the large N_c limit, all of the higher partial waves vanish, and the pion coupling to baryons is purely p-wave, isospin one, i.e. it has $I = J$. Mattis et al. [8] originally derived the $I = J$ rule in the Skyrme model. The $I = J$ rule is true in large N_c QCD because G^{ia}/N_c which has $I = J = 1$ is of order one, whereas the matrix elements of I/N_c and J/N_c, which have $|I - J| = 1$, are

of order $1/N_c$. We also get the additional result that the higher partial wave pion-couplings are of order $1/N_c^{\ell-2}$ in the $1/N_c$ expansion. In general, pion couplings which violate the $I=J$ rule are suppressed by a factor of $1/N_c^{|I-J|}$ relative to the $I=J$ coupling, which is of order N_c. A similar result also holds for other meson-baryon couplings. For example, p-wave kaon couplings (which have $J=1$ and $I=1/2$) are of order $\sqrt{N_c}$, and p-wave η couplings (which have $J=1$ and $I=0$) are of order one [3]; the spin-one ρ coupling and spin-zero ω couplings are of order N_c; and the spin-zero ρ coupling and spin-one ω coupling are of order one [8,10].

2. Three Flavors

The analysis of the $1/N_c$ expansion of the axial current for three or more flavors is essentially the same. We employ the familiar language of $SU(3)$ symmetry here for concreteness.

For three flavors, matrix elements of the flavor generators T^a and matrix elements of the spin-flavor generators G^{ia} are not the same order in the $1/N_c$ expansion everywhere in the flavor weight diagram. Consider, for example, the weight diagram of the spin-1/2 baryons (Fig. 2). Baryons with strangeness of order N_c^0 (near the top of the weight diagram) have matrix elements of T^a, $a=1,2,3$ and G^{i8} of order one; matrix elements of T^a and G^{ia}, $a=4,5,6,7$, of order $\sqrt{N_c}$; and matrix elements of T^8 and G^{ia}, $a=1,2,3$ of order N_c. In other regions of the weight diagram, matrix elements of different linear combinations of the T's and G's are of order N_c, $\sqrt{N_c}$ and one. This non-trivial N_c-dependence of the matrix elements of T^a and G^{ia} makes the analysis of the $1/N_c$ expansion for three (or more) flavors more complicated than that for two flavors, because matrix elements of the flavor generators T^a are not suppressed relative to G^{ia}. In fact, matrix elements of T^a can be a factor of N_c larger than matrix elements of G^{ia} (for some values of the index a) in some regions of the flavor weight diagrams.

The $1/N_c$ expansion of the axial vector current is tractable for three or more flavors because of the operator reduction rule and because matrix elements of the spin J are suppressed for our choice of limit. The general form of the $1/N_c$ expansion for the axial current eq. (7.2) contains terms with arbitrary powers of T^a/N_c, G^{ia}/N_c and J^i/N_c. Factors of T^a/N_c and G^{ia}/N_c are of order one somewhere in the weight diagram, but J^i/N is of order $1/N_c$ everywhere. Thus, operators with arbitrary powers of T^a/N_c and G^{ia}/N_c are all equally important and must be retained. The operator reduction rule, however, allows us to reduce this set of operators to the subset with one factor of T^a or G^{ia}, eq. (9.7). The operator expansion eq. (9.7) can be truncated after the first two terms,

$$A^{ia} = a_1\ G^{ia} + b_2\ \frac{1}{N_c}\mathcal{D}_2^{ia} = a_1\ G^{ia} + b_2 \frac{J^i T^a}{N_c}, \tag{9.9}$$

since all other terms in eq. (9.7) are suppressed by at least $1/N_c^2$ relative to the terms which have been retained *everywhere* in the weight diagram. From the operator definitions eqs. (9.1)–(9.6), it is easy to see that: (i) The operators $\mathcal{O}_n^{ia}/N_c^{n-1}$, n odd, are of order $1/N_c^{n-1}$ relative to G^{ia} everywhere in the weight diagram; (ii) The operators $\mathcal{D}_n^{ia}/N_c^{n-1}$, n odd, are suppressed by $1/N^{n-1}$ relative to G^{ia} everywhere in the weight diagram; and (iii) The operators $\mathcal{D}_n^{ia}/N_c^{n-1}$, n even, are suppressed by $1/N^{n-2}$ relative to $\mathcal{D}_2^{ia}/N_c$ everywhere in the weight diagram. Although the second term in eq. (9.9) contains an explicit factor of $1/N_c$, this term is not necessarily suppressed relative to the first term, and must be retained for a valid truncation.

In eq. (9.3) of ref. [3], we defined the $SU(3)$-invariant meson couplings $\mathcal{M}$ and $\mathcal{N}$, and proved that

$$\frac{\mathcal{N}}{\mathcal{M}} = \frac{1}{2} + \frac{\alpha}{N_c} + \mathcal{O}\left(\frac{1}{N_c^2}\right). \tag{9.10}$$

Eq. (9.9) implies

$$\alpha = -\frac{3}{2}\left(1+\frac{b_2}{a_1}\right), \tag{9.11}$$

so that b_2/a_1 determines the parameter α. Eq. (9.9) is the quark representation analog of eq. (9.3) of ref. [3] for the meson couplings in the $SU(3)$ limit.

We are interested in the baryon couplings for the physical case of $N_c=3$. In the $SU(3)$ limit, the octet baryon couplings are described by the parameters D and F, the decuplet to octet transition couplings are described by $\mathcal{C}$, and the decuplet couplings are described by $\mathcal{H}$ [19]. These parameters can be determined by taking matrix elements of eq. (9.9) in quark model baryon states,

$$\begin{aligned} D &= \frac{1}{2}a_1, \\ F &= \frac{1}{3}a_1 + \frac{1}{6}b_2, \\ \mathcal{C} &= -a_1, \\ \mathcal{H} &= -\frac{3}{2}\left(a_1+b_2\right), \end{aligned} \tag{9.12}$$

where we have set $N_c=3$. Note that the diagonal operator $\mathcal{D}_2$, which is the product of spin and flavor generators, only affects the diagonal couplings F and $\mathcal{H}$. The parameter b_2 produces deviations from the $SU(6)$ prediction. One can eliminate the unknown parameters to get the two relations,

$$\mathcal{C} = -2D, \qquad \mathcal{H} = 3D - 9F. \tag{9.13}$$

The first relation is an $SU(6)$ relation. The F/D ratio is given by

$$\frac{F}{D} = \frac{2}{3} + \frac{b_2}{3a_1}. \tag{9.14}$$

One cannot determine the accuracy in $1/N_c$ of F/D from eq. (9.14), since this equation is only valid at $N_c=3$.

One can also analyze the meson couplings for arbitrary N_c, and study the limit as $N_c \to 3$. The $1/N_c$ dependence of meson-baryon couplings depends on the location of the baryon in the $SU(3)$ weight diagram. If one considers baryons with strangeness of order N_c^0, the second term in eq. (9.9) is of order $1/N_c^2$ relative to the first term for $a = 1, 2, 3$, (π couplings); of relative order $1/N_c$ for $a = 4, 5, 6, 7$, (K couplings); and of the same order for $a = 8$ (η couplings). For baryons with strangeness of order N_c^0, the unknown ratio b_2/a_1 affects pion couplings at order $1/N_c^2$, kaon couplings at order $1/N_c$, and η couplings at order one [3]. If one defines F/D using ratios of pion couplings between baryons of strangeness N_c^0, one concludes that $F/D = 2/3 + \mathcal{O}(1/N_c^2)$. Similarly a determination using K and η couplings gives $F/D = 2/3 + \mathcal{O}(1/N_c)$ and $F/D = 2/3 + \mathcal{O}(1)$, respectively. The ratio of π couplings was used in ref. [3] since it gives the most accurate determination of F/D. For a more detailed discussion of the meson-couplings in the $SU(3)$ limit and the F/D ratio, see Sections IX and XI of ref. [3].

At next order in the $1/N_c$ expansion, the meson couplings have the form

$$A^{ia} = a_1\, G^{ia} + b_2 \frac{J^i T^a}{N_c} + b_3 \frac{\mathcal{D}_3^{ia}}{N_c^2} + c_3 \frac{\mathcal{O}_3^{ia}}{N_c^2}. \qquad (9.15)$$

Since there are four linearly independent operators in eq. (9.15), the four parameters D, F, $\mathcal{C}$ and $\mathcal{H}$, are all independent at this order in $1/N_c$ expansion, and there is no prediction.

B. Meson Couplings with Perturbative SU(3) Breaking

In this section, we compute the $1/N_c$ expansion for a $(1, 8)$ operator (such as the axial current or meson coupling) to first order in $SU(3)$ breaking. Flavor $SU(3)$ breaking in QCD is due to the light quark masses and transforms as a flavor octet. We will neglect isospin breaking, and work to linear order in the $SU(3)$ symmetry breaking perturbation $\epsilon\mathcal{H}^8$.

The $SU(3)$ symmetry-breaking correction to the axial current is computed to linear order in $SU(3)$ symmetry breaking from the tensor product of the axial current which transforms as $(1, 8)$, and the perturbation $(0, 8)$. This tensor product contains a $(1, 0)$, $(1, 8)$, $(1, 8)$ $(1, 10 + \overline{10})$, and $(1, 27)$. The form of these operators is determined by a straightforward application of the operator reduction rule.

The series $\mathcal{O}_n$ for the $(1, 0)$ operators starts with the one-body operator $\mathcal{O}_1 = J^i$. Consecutive terms in the series are generated by $\mathcal{O}_{n+2} = \left\{J^2, \mathcal{O}_n\right\}$. The higher order terms in the expansion of the meson couplings are all suppressed by at least $1/N_c^2$ relative to the leading operator J^i, which gives the symmetry breaking contribution

$$\delta A_1^{ia} \propto \delta^{a8} J^i\ . \qquad (9.16)$$

The $(1, 8)$ operator has the same form as eq. (9.9). The $(1, 8)$ $SU(3)$-breaking correction to the axial currents is of the form

$$\delta A_8^{ia} \propto d^{ab8} \left[c_1\, G^{ib} + c_2 \frac{J^i T^b}{N_c}\right]\ . \qquad (9.17)$$

A similar term with the d-symbol replaced by an f-symbol is forbidden by time reversal. Neglected operators are suppressed by $1/N_c^2$ relative to the operators we have retained.

There is a second $(1, 8)$ operator of the form

$$\delta A_8^{ia} \propto f^{ab8} \epsilon^{ijk} \{J^j, G^{kc}\}\ . \qquad (9.18)$$

Neglected operators contain additional factors of J^2/N_c^2, and are suppressed by at least $1/N_c^2$ relative to eq. (9.18). The operator eq. (9.18) can be written as

$$\delta A_8^{ia} \propto \left[J^2, \left[T^8, G^{ia}\right]\right], \qquad (9.19)$$

which shows that it contributes only to amplitudes which change both J and strangeness (T^8).

The operator reduction rule implies that the $(1, 10+\overline{10})$ $((1, \bar{a}s+\bar{s}a))$ representation is given by (see Table IX and Appendix A)

$$\{G^{ia}, T^b\} - \{G^{ib}, T^a\} - \frac{2}{3} f^{abc} f^{cgh} \{G^{ig}, T^h\}. \qquad (9.20)$$

Additional operators have at least two more factors of J, and are suppressed by $1/N_c^2$ relative to the operator in eq. (9.20). Note that

$$\left[J^2, \left[T^b, G^{ia}\right]\right] = \left[T^2, \left[T^b, G^{ia}\right]\right] = f^{abc} f^{cgh} \{G^{ig}, T^h\}, \qquad (9.21)$$

where the first equality follows from eq. (5.12), which implies that $J^2 - T^2$ equals a constant. Substituting eq. (9.21) into eq. (9.20) gives

$$\delta A_{10+\overline{10}}^{ia} \propto \{G^{ia}, T^8\} - \{G^{i8}, T^a\} - \frac{2}{3}\left[J^2, \left[T^8, G^{ia}\right]\right]\ . \qquad (9.22)$$

The $(1, 27)$ symmetry-breaking correction can be built out of the spin-zero 27 in $\{T^a, T^b\}$; the spin-one 27 in $\{G^{ia}, T^b\} + \{G^{ib}, T^a\}$; and the spin-two 27 in $\{G^{ia}, G^{ib}\} + \{G^{ia}, G^{ib}\}$ (see Table IX). The spin-zero and spin-two 27's must be combined with additional factors of J/N_c to form a $(1, 27)$, so to leading order in $1/N_c$, only the spin-one 27 needs to be retained. We do not need to subtract off the singlet and octet parts of $\{G^{ia}, T^b\} + \{G^{ib}, T^a\}$ because they can be absorbed into the symmetry-breaking singlet and octet operators (9.16) and (9.17) which have already been included. Thus, the symmetry-breaking $(1, 27)$ correction is proportional to

$$\delta A^{ia}_{27} \propto \{G^{ia}, T^8\} + \{G^{i8}, T^a\} \ . \tag{9.23}$$

The neglected higher order 27 operators are only suppressed by $1/N_c$ relative to the operator we have retained.

The leading order $1/N_c$ expansion for the axial current to first order in $SU(3)$ symmetry-breaking is given by the lowest order $SU(3)$ symmetry result eq. (9.9) and the sum of the perturbations eqs. (9.16)–(9.23). The perturbations involving T^8 and G^{i8} can be simplified using eq. (8.2). The final expression for the axial current to linear order in symmetry breaking is

$$\begin{aligned} A^{ia} + \delta A^{ia} &= \left(a + \epsilon c_1 d^{ab8}\right) G^{ib} + \left(b + \epsilon c_2 d^{ab8}\right) \frac{J^i T^b}{N_c} \\ &+ \epsilon c_3 \frac{\left\{G^{ia}, N_s\right\}}{N_c} + \epsilon c_4 \frac{\left\{J^i_s, T^a\right\}}{N_c} \\ &+ \epsilon \frac{c_5}{3N_c} \left[J^2, \left[N_s, G^{ia}\right]\right] + \epsilon c_6 \delta^{a8} J^i, \end{aligned} \tag{9.24}$$

where the parameter ϵ emphasizes which terms result from symmetry breaking, and the parameters a and b contain terms of order ϵ from the substitutions for T^8 and G^{i8}. The double commutator term in eq. (9.24) only contributes to processes which change both spin and strangeness.

The coefficient c_6 in eq. (9.24) can be constrained due to the following considerations. We focus on baryons containing no strange quarks. The matrix elements of the isovector axial current ($\bar{q}\gamma^\mu\gamma_5\tau^a q$) are of order N_c in the strangeness zero sector, since the expansion for the isovector axial current involves the one-body operator G^{ia}, which has matrix elements of order N_c. The flavor singlet axial current ($\bar{q}\gamma^\mu\gamma_5 q$) has matrix elements of order one in the strangeness zero sector, since matrix elements of the one-body operator J^i are of order one. The expansion of the strange axial current ($\bar{s}\gamma^\mu\gamma_5 s$) involves the one-body operator J^i, so naively one expects the matrix elements of the strange axial current also to be of order one. However, in the strangeness-zero sector, $\bar{s}\gamma^\mu\gamma_5 s$ can only couple to the baryon through a closed s-quark loop, which is accompanied by a $1/N_c$ suppression factor. Thus, matrix elements of the strange axial current are of order $1/N_c$, not order one, for strangeness-zero baryons. $SU(3)$ breaking effects in the strangeness-zero sector involve closed s-quark loops, and so $SU(3)$ breaking has a $1/N_c$ suppression factor. Diagrams involving a closed loop, such as those in Fig. (6), lead to quark mass dependence in axial current matrix elements. Graphs, such as in Fig. (6a), in which the axial current is not inserted in the closed loop depend only on $\mathrm{Tr}M$, where M is the quark mass matrix, and yield matrix elements which are quark mass-dependent, but which do not violate $SU(3)$. (An example of this kind is the pion-nucleon sigma term.) $SU(3)$ violation arises from diagrams such as Fig. (6b) in which the axial current operator is inserted in the closed quark loop. This diagram can only produce $SU(3)$ violation in the baryon axial currents at order $1/N_c$. Thus, $SU(3)$ violation in the axial currents must be order $1/N_c$ for strangeness-zero baryons.

Imposing this constraint on the expansion eq. (9.24) fixes the coefficient c_6. In the strangeness zero sector, eq. (9.24) reduces to

$$\left(a + \epsilon c_1 d^{ab8}\right) G^{ib} + \left(b + \epsilon c_2 d^{ab8}\right) \frac{J^i T^b}{N_c} + \epsilon c_6 \delta^{a8} J^i. \tag{9.25}$$

Requiring that there is no $SU(3)$ symmetry breaking between the π and η couplings at order one gives the constraint

$$3c_6 = c_1 + c_2, \tag{9.26}$$

which reduces the number of parameters in eq. (9.24) to six parameters. Eqs. (9.24) and (9.26) yield the following expressions for the pion, kaon and η couplings.

The pion couplings are given by

$$\begin{aligned} \pi^{ia} &= \tilde{a} G^{ia} + \tilde{b} \frac{J^i T^a}{N_c} \\ &+ \epsilon c_3 \frac{\left\{G^{ia}, N_s\right\}}{N_c} + \epsilon c_4 \frac{\left\{J^i_s, T^a\right\}}{N_c}, \end{aligned} \tag{9.27}$$

where

$$\tilde{a} = a + \frac{1}{\sqrt{3}} \epsilon c_1, \quad \tilde{b} = b + \frac{1}{\sqrt{3}} \epsilon c_2.$$

If one considers baryons with strangeness of order N_c^0, the matrix elements of G^{ia} are of order N_c, whereas the matrix elements of $J^i T^a$ and $J^i_s T^a$ are of order unity. To the order we are working, it is consistent to retain only

$$\pi^{ia} = \tilde{a} G^{ia} + \epsilon c_3 \frac{\left\{G^{ia}, N_s\right\}}{N_c}. \tag{9.28}$$

Eq. (9.28) determines the pion couplings of baryons with a given strangeness to relative order $1/N_c^2$, and it implies that pion couplings have a linear dependence on strangeness at relative order $1/N_c$ [3].

The kaon couplings are given by

$$\begin{aligned} K^{ia} &= \left(\tilde{a} - \frac{\sqrt{3}}{2} \epsilon c_1\right) G^{ia} + \left(\tilde{b} - \frac{\sqrt{3}}{2} \epsilon c_2\right) \frac{J^i T^a}{N_c} \\ &+ \epsilon c_3 \frac{\left\{G^{ia}, N_s\right\}}{N_c} + \epsilon c_4 \frac{\left\{J^i_s, T^a\right\}}{N_c} \\ &+ \epsilon \frac{c_5}{3N_c} \left[J^2, \left[N_s, G^{ia}\right]\right], \end{aligned} \tag{9.29}$$

where $\left[N_s, G^{ia}\right] = \pm G^{ia}$ depending on whether one is looking at terms which annihilate K^+, K^0 or $K^-, \bar{K}^0$. The double commutator term only contributes to processes which change J^2.

The η couplings are given by

$$\eta^i = \frac{1}{2\sqrt{3}}\left(\tilde{a}+\tilde{b}\right)J^i + \left(-\frac{\sqrt{3}}{2}\tilde{a}+\epsilon c_1+\frac{1}{\sqrt{3}}\epsilon c_4\right)J_s^i$$
$$+\left(-\frac{\sqrt{3}}{2}\tilde{b}+\epsilon c_2+\frac{1}{\sqrt{3}}\epsilon c_3\right)\frac{N_s}{N_c}J^i$$
$$-\sqrt{3}\epsilon\left(c_3+c_4\right)\frac{N_s}{N_c}J_s^i. \qquad (9.30)$$

A detailed comparison of these equations with the experimental data is given in ref. [12]. They provide a good description of the data on decuplet → octet decays (such as $\Delta \to N\pi$), and baryon semileptonic decays.

C. Meson Couplings Without $SU(3)$ Symmetry

The $1/N_c$ expansion can also be used to obtain meson couplings without assuming $SU(3)$ symmetry. For the $1/N_c$ expansion to be valid, one must work with states which differ in strangeness by order unity. We will work near the top of the $SU(3)$ weight diagram, where the number of strange quarks in the baryon is of order one, and take the large-N_c limit with the number of strange quarks of the baryon held fixed. This form of the $1/N_c$ expansion was discussed in detail in ref. [3]. We rederive the results obtained there using the quark representation, and then compare with the perturbative symmetry breaking results obtained in the previous subsection.

1. Pion Couplings

The isovector axial current and p-wave pion couplings have $(J,I)_S$ quantum numbers $(1,1)_0$. The second operator reduction rule implies that all t, $t^\dagger$, Y, and $Y^\dagger$ operators can be eliminated using the identities, so the pion coupling can be written as a function of G^{ia}, I^a, J^i (or J^i_{ud}), J^i_s and N_s. Operators with contracted isospin indices can be eliminated, so the operators which remain have either one G^{ia} and no I's or one I^a and no G's. There are five operator series with the correct time-reversal properties. They are given by the operators

$$\begin{aligned}
&G^{ia}\ ,\\
&\frac{1}{N_c}J^i_{ud}I^a\ ,\\
&\frac{1}{N_c}J^i_sI^a\ ,\\
&\frac{1}{N_c^2}\left\{J^i_{ud},\left\{G^{ka},J^k_s\right\}\right\}\ ,\\
&\frac{1}{N_c^2}\left\{J^i_s,\left\{G^{ka},J^k_s\right\}\right\}\ ,
\end{aligned} \qquad (9.31)$$

times polynomials

$$\mathcal{P}\left(\frac{N_s}{N_c},\frac{J_{ud}^2}{N_c^2},\frac{J_{ud}\cdot J_s}{N_c^2}\right) \qquad (9.32)$$

in the arguments N_s/N_c, J^2_{ud}/N_c^2, and $J_{ud}\cdot J_s/N_c^2$. Each operator series involves a different polynomial. Once the operator series has been determined, it is more convenient to rewrite the polynomials as functions of N_s/N_c, J^2/N_c^2 and I^2/N_c^2, using $I^2 = J^2_{ud}$ and $J^2=(J_{ud}+J_s)^2$.

For baryons with strangeness of order unity, the matrix elements of G^{ia} are order N_c, and the matrix elements of I, J (or J_{ud}), and J_s are order one, so the dominant operator series is the series generated by G^{ia}. The four other operators in eq. (9.31) are suppressed by a factor of $1/N_c^2$ relative to G^{ia}. The first operator series contains the operators G^{ia} and N_sG^{ia}/N_c up to terms of relative order $1/N_c^2$ compared to the leading operator G^{ia}. Thus, eq. (9.31) produces the same result as the perturbative breaking formula eq. (9.28) to relative order $1/N_c^2$. The derivation of this formula using only $SU(2)\times U(1)$ flavor symmetry implies that the equal spacing rule for pion-baryon couplings is valid to all orders in $SU(3)$ symmetry breaking [3].

2. Kaon Couplings

The kaon couplings transform as $(1,1/2)_1$ and contain either one t and no Y, or one Y and no t. There are six basic operator series which contribute; they are generated by the operators

$$\begin{aligned}
&Y^{i\alpha}\ ,\\
&\frac{1}{N_c}\left\{t^\alpha,J^i_{ud}\right\}\ ,\\
&\frac{1}{N_c}\left\{t^\alpha,J^i_s\right\}\ ,\\
&\frac{1}{N_c}i\epsilon^{ijk}\left\{Y^{j\alpha},J^k_{ud}\right\}\ ,\\
&\frac{1}{N_c^2}\left\{J^i_{ud},\left\{Y^{k\alpha},J^k_{ud}\right\}\right\}\ ,\\
&\frac{1}{N_c^2}\left\{J^i_s,\left\{Y^{k\alpha},J^k_{ud}\right\}\right\}\ ,
\end{aligned} \qquad (9.33)$$

times polynomials of N_s/N_c, I^2/N_c^2 and J^2/N_c^2.

For baryons with strangeness of order unity, the matrix elements of $Y^{i\alpha}$ and t^α are order $\sqrt{N_c}$. Thus, the leading order operator for the kaon couplings is $Y^{i\alpha}$. There are four additional operators which contribute at relative order $1/N_c$. They are $\{N_s,Y^{i\alpha}\}/N_c$ and the three operators proportional to $1/N_c$ in eq. (9.33).

The perturbative $SU(3)$ breaking expansion eq. (9.29) has the same structure outlined above. The commutation relations in Table X imply that the double commutator term in eq. (9.29) is equal to $i\epsilon^{ijk}\left\{Y^{j\alpha},J^k\right\}$. This operator can be rewritten in terms of J_{ud} and J_s. The piece involving J_s reduces to a linear combination of the other operators in eq. (9.29) by the operator identities. Thus, eq. (9.29) contains the same five operators as the completely broken analysis to relative order $1/N_c^2$. The perturbative breaking formula determines four of the five

kaon coefficients in terms of the coefficients for the π and η couplings. This relationship is lost for completely broken $SU(3)$ symmetry.

3. Eta Couplings

The isoscalar axial current transforms as $(1,0)_0$. The second operator reduction rule implies that the general expansion is generated by the operators

$$\begin{aligned} &J^i_{ud} \ , \\ &J^i_s \ , \end{aligned} \tag{9.34}$$

times polynomials of N_s/N_c, I^2/N_c^2 and J^2/N_c^2. Eq. (9.30) gives the same expansion up to terms of order $1/N_c^2$ as eq. (9.34). There is no relationship between the π and η coefficients in eq. (9.28) and eq. (9.30) for perturbatively broken or for completely broken $SU(3)$ symmetry.

X. BARYON MASSES

In this section, we study the baryon masses in the flavor $SU(3)$ limit, to first order in $SU(3)$ breaking, and for $SU(2)\times U(1)$ flavor symmetry.

A. Baryon Masses in the Flavor Symmetry Limit

The $1/N_c$ expansion for the baryon masses in the $SU(F)$ flavor symmetry limit can be obtained using using the operator reduction rule. The general form of the quark operator expansion of the baryon Hamiltonian is given by eq. (7.2). The Hamiltonian is a spin and flavor singlet. The expansion involves the zero-body operator $\mathbb{1}$ and polynomials in the one-body operators J^i, T^a and G^{ia} which transform as the $(0,0)$ representation of $SU(2)\times SU(F)$. To obtain a flavor singlet, all flavor indices on the T's and G's must be contracted using $SU(F)$ tensors with adjoint indices, which can be written as products of the $SU(F)$ invariant tensors δ^{ab}, d^{abc} and f^{abc}. All such objects can be removed by the operator reduction rule. Thus, the Hamiltonian can be written purely as an expansion in J^i, and, by rotation invariance, it can only be a function of J^2. Thus, the baryon mass operator is given by

$$M = N_c\ \mathcal{P}\left(\frac{J^2}{N_c^2}\right) \tag{10.1}$$

where $\mathcal{P}$ is a polynomial. This result reproduces the form of the baryon mass expansion obtained previously [2,3,10,11].

B. Baryon Masses with Perturbative $SU(3)$ Breaking

Flavor $SU(3)$ symmetry is broken because the light quarks have different masses. The perturbation transforms as the $(0,adj)$ irreducible representation of $SU(2)\times SU(F)$. The dominant $SU(3)$ breaking transforms as the eighth component of a flavor octet. Isospin breaking effects are much smaller, and will be neglected here.

The quark operator expansion for a $(0,adj)$ QCD operator is of the form given in eq. (7.2). The operator reduction rule implies that only n-body operators with a single factor of either T^a or G^{ia} need to be retained. There is only one one-body operator,

$$\mathcal{O}_1^a = T^a, \tag{10.2}$$

and there is only one two-body operator,

$$\mathcal{O}_2^a = \left\{J^i, G^{ia}\right\}, \tag{10.3}$$

allowed by the operator reduction rule. In general, there is only one independent n-body operator for each n. All of these operators can be generated recursively from operators $\mathcal{O}_1^a$ and $\mathcal{O}_2^a$ by anticommuting with J^2,

$$\mathcal{O}_{n+2}^a = \left\{J^2, \mathcal{O}_n^a\right\}\ . \tag{10.4}$$

The set of operators $\mathcal{O}_n^a$, $n=1,2,\ldots,N_c$, forms a complete set of linearly independent spin-zero adjoints. Thus, the flavor symmetry breaking component of the Hamiltonian has the expansion

$$\mathcal{H}^a = \sum_{n=1}^{N_c} b_n \frac{1}{N_c^{n-1}} \mathcal{O}_n^a, \tag{10.5}$$

where b_n are unknown coefficients. Since J is of order one, the contribution of $\mathcal{O}_{n+2}$ to the baryon mass in eq. (10.5) is suppressed by $1/N_c^2$ relative to that of $\mathcal{O}_n$. Thus, the expansion of the symmetry breaking perturbation can be truncated after the first two terms, up to corrections of relative order $1/N_c^2$. The expansion for the baryon masses, including $SU(3)$ breaking perturbatively to linear order, is

$$M = a_0 N_c\ + a_2 \frac{J^2}{N_c} + \epsilon b_1 T^8 + \epsilon \frac{1}{N_c} b_2 \left\{J^i, G^{i8}\right\} \tag{10.6}$$

up to terms of order $1/N_c^2$. An explicit factor of ϵ appears in front of the last two terms in eq. (10.6) to emphasize which terms arise from symmetry breaking. Note that ϵ should not be regarded as an additional parameter, since b_1 and b_2 are unknowns. The general expansion for the masses to linear order in symmetry breaking has the form

$$\begin{aligned} M =&\ N_c\ \mathcal{P}_0\left(\frac{J^2}{N_c^2}\right) + \epsilon \mathcal{P}_1\left(\frac{J^2}{N_c^2}\right) T^8 \\ &+ \epsilon \frac{1}{N_c} \mathcal{P}_2\left(\frac{J^2}{N_c^2}\right)\left\{J^i, G^{i8}\right\}\ , \end{aligned} \tag{10.7}$$

where $\mathcal{P}_i$ are arbitrary polynomials in their argument.

Eq. (10.6) can be rewritten using the substitutions (8.2) for T^8 and G^{i8},

$$M = a_0 N_c \; + a_2 \frac{J^2}{N_c} + \epsilon b_1 N_s + \epsilon \frac{1}{N_c} b_2 \left\{ J^i, J_s^i \right\}, \quad (10.8)$$

where pieces of the last two terms in eq. (10.8) have been reabsorbed into the first two terms. Note that

$$\left\{ J^i, J_s^i \right\} = J^2 + J_s^2 - I^2 \quad (10.9)$$

since $J^i - J_s^i = J_{ud}^i$ and $J_{ud}^2 = I^2$. The masses of the eight isomultiplets of the spin-1/2 octet and spin-3/2 decuplet baryons are written in terms of four unknown parameters in eq. (10.6), so there are four mass relations which are satisfied to linear order in symmetry breaking and to order $1/N_c^2$ in the $1/N_c$ expansion,

$$\begin{aligned}
&\frac{1}{3}\left(\Sigma + 2\Sigma^*\right) - \Lambda = \frac{2}{3}\left(\Delta - N\right), && (10.10)\\
&\Sigma^* - \Sigma = \Xi^* - \Xi, && (10.11)\\
&\frac{3}{4}\Lambda + \frac{1}{4}\Sigma - \frac{1}{2}\left(N + \Xi\right) = 0, && (10.12)\\
&\left(\Sigma^* - \Delta\right) = \left(\Xi^* - \Sigma^*\right), && (10.13)\\
&\left(\Xi^* - \Sigma^*\right) = \left(\Omega - \Xi^*\right), && (10.14)
\end{aligned}$$

where only four of the above five relations are linearly independent. The first two relations are spin-flavor relations. The third relation is the Gell-Mann–Okubo formula for the baryon octet, and the last two relations are the equal spacing rule of the decuplet. The Gell-Mann–Okubo formula and the equal spacing rule are consequences of $SU(3)$ symmetry alone, but the other two relations connect mass splittings in the octet and decuplet, and depend on the spin-flavor structure of the mass splittings.

C. Baryon Masses with Completely Broken $SU(3)$ Symmetry

The analysis of the baryon masses can be performed using only $SU(2) \times U(1)$ flavor symmetry. Such an analysis yields baryon mass relations which are valid to all orders in $SU(3)$ breaking even for large nonperturbative $SU(3)$ breaking. The completely broken $SU(3)$ analysis constructs a $1/N_c$ expansion of all quark operators transforming as spin, isospin and strangeness singlets, $(0,0)_0$. Application of the second operator reduction rule yields the mass expansion,

$$M = a_0 N_c \; + a_1 N_s + a_{21} \frac{J^2}{N_c} + a_{22} \frac{I^2}{N_c} + a_{23} \frac{N_s^2}{N_c}, \quad (10.15)$$

which is valid up to relative order $1/N_c^3$. Eq. (10.15) yields three linearly independent mass relations. Relations (10.10) and (10.11) are still valid. The last three relations are replaced by

$$\begin{aligned}
&\frac{3}{4}\Lambda + \frac{1}{4}\Sigma - \frac{1}{2}\left(N + \Xi\right) = -\frac{1}{4}\left(\Omega - \Xi^* - \Sigma^* + \Delta\right),\\
&\frac{1}{2}\left(\Sigma^* - \Delta\right) - \left(\Xi^* - \Sigma^*\right) + \frac{1}{2}\left(\Omega - \Xi^*\right) = 0. \qquad (10.16)
\end{aligned}$$

The first relation relates breaking of the Gell-Mann–Okubo formula to breaking of one linear combination of the two equal spacing rule relations. The second relation is the other linear combination of the two equal spacing rule relations. The above results were derived previously in ref. [3], and the reader is referred to this work for additional discussion. Comparison of eq. (10.15) and the perturbative formula with linear symmetry breaking eq. (10.8) shows that the completely broken case replaces the operator $\left\{ J^i, J_s^i \right\}$ by two independent operators, I^2 and N_s^2.

The general form of the mass expansion for completely broken flavor $SU(3)$ symmetry to all orders in the $1/N_c$ expansion is

$$M = N_c \; \mathcal{P}\left(\frac{N_s}{N_c}, \frac{J^2}{N_c^2}, \frac{I^2}{N_c^2}\right), \quad (10.17)$$

where $\mathcal{P}$ is an arbitrary polynomial in its arguments.

XI. MAGNETIC MOMENTS

The baryon magnetic moments were analyzed in ref. [4] using the $1/N_c$ expansion in the Skyrme representation. In this section, we compare those results with the analysis in the quark representation.

The baryon magnetic moments transform as $(1,8)$ under $SU(2) \times SU(3)$, so the expansions derived in Section IX for the axial couplings can be applied to the magnetic moments as long as one remembers that the coefficients of the operators in the $1/N_c$ expansion can have different values for the magnetic moments than for the axial couplings. The magnetic moments are proportional to the quark charge matrix $\mathcal{Q} = \mathrm{diag}(2/3, -1/3, -1/3)$, so they can be separated into isovector and isoscalar components. The analysis of Section IX showed that the $1/N_c$ expansions of the isovector and isoscalar axial currents are unrelated for perturbatively broken $SU(3)$ symmetry, and that the same expansions are obtained for perturbatively broken and completely broken $SU(3)$ symmetry. Thus, the analysis of the magnetic moments can be restricted to the cases of exact $SU(3)$ symmetry and completely broken $SU(3)$ symmetry.

The analysis of the magnetic moments in the $SU(3)$ symmetry limit is completely analogous to the analysis of the meson couplings in the flavor symmetry limit, and will not be repeated here. The magnetic moments for $N_c = 3$ are parametrized by four $SU(3)$ invariants, μ_D, μ_F, $\mu_{\mathcal{C}}$ and $\mu_{\mathcal{H}}$ [21]. These parameters satisfy equations which are the analogues of eqs. (9.12–9.14) for the meson couplings D, F, $\mathcal{C}$, and $\mathcal{H}$.

The isovector and isoscalar magnetic moments for completely broken $SU(3)$ symmetry are given by the operator

expansions eqs. (9.31) and eq. (9.34). Retaining terms to order $1/N_c^2$ in these expansions yields

$$\mu^{i3} = c_1 G^{i3} + c_2 \frac{N_s}{N_c} G^{i3}, \tag{11.1}$$
$$\mu^{i8} = d_1 J^i + d_2 J_s^i + d_3 \frac{N_s}{N_c} J^i + d_4 \frac{N_s}{N_c} J_s^i,$$

which is precisely the same expansion used in ref. [4], with the quark operator G^{ia} of the quark representation replacing the analogous Skyrme representation operator $N_c X^{ia}$.

The eight isoscalar magnetic moment relations $S1$–$S8$ of ref. [4] hold without change, since the matrix elements of J^i and J_s^i are identical in the Skyrme and quark representations. All eight relations are valid in the non-relativistic quark model. Relations $S1$–$S6$ are valid to order $1/N_c^2$. Only one of these relations, $S1$, is measured experimentally; it holds to $4 \pm 5\%$. Relation $S7$,

$$5(p+n) - (\Xi^0 + \Xi^-) = 4(\Sigma^+ + \Sigma^-) \ , \tag{11.2}$$

is true in large-N_c QCD at leading order, but is violated at order $1/N_c$. It works experimentally to $22 \pm 4\%$. Relation $S8$,

$$(p+n) - 3\Lambda = \frac{1}{2}(\Sigma^+ + \Sigma^-) - (\Xi^0 + \Xi^-) \ , \tag{11.3}$$

is true in large-N_c QCD at leading order. It is also valid at order $1/N_c$ if one imposes $SU(3)$ symmetry in the $1/N_c$ terms, without imposing $SU(3)$ symmetry in the leading order terms. (This restriction corresponds to neglecting the order ϵ terms in $J^i N_s/N_c$ and $J_s^i N_s/N_c$ in eq. (9.30), but retaining them in J^i and J_s^i, which eliminates the $J^i N_s/N_c$ operator from the expansion.) Thus, the violation of this relation is suppressed by $SU(3)$ symmetry breaking/N_c. Relation $S8$ holds experimentally to $7 \pm 1\%$. Thus, the $1/N_c$ expansion provides some understanding of why one quark model relation works better than the others. This example also shows that the $1/N_c$ expansion is not the same as the non-relativistic quark model, even in the quark representation.

Ten isovector relations were derived in ref. [4]. Relations $V1$–$V7$ hold without change in the quark representation to relative order $1/N_c^2$. Relations $V8$–$V10$ had slightly different forms in the non-relativistic quark model, and in the $1/N_c$ expansion using the Skyrme representation. The isovector relations derived using eq. (11.1) of the quark representation are identical to those in the quark model, i.e. one obtains relations $V8_2$, $V9_2$ and $V10_4$. Relations $V8_2$ and $V9_2$ are valid to relative order $1/N_c^2$ using the quark representation. Relation $V10_4$ is valid at leading order in the $1/N_c$ expansion and at first subleading order in the $1/N_c$ expansion in the $SU(3)$ limit using the quark representation, and is the analogue of relations $V10_2$ and $V10_3$ using the Skyrme representation. The difference between the Skyrme and quark representation forms of the isovector relations is due to a difference of order $1/N_c^2$ in the matrix elements of the two different representations. Since we have neglected terms of order $1/N_c^2$, the difference between the results is not significant to this order. In particular, one of the conclusions of ref. [4], that the $1/N_c$ expansion gives a better prediction than the quark model for the $\Delta^+ \to p\gamma$ transition moment, is incorrect; the difference in the two predictions is an order $1/N_c^2$ effect.

The relation S/V_1 between the isoscalar and isovector magnetic moments is valid in both the Skyrme and quark representations in the $SU(3)$ limit to two orders in $1/N_c$. Note that the matrix element of G^{ia} in the proton is $(N_c+2)/12$ for our definition of G^{ia}.

Finally, note that eq. (9.29) can be used to obtain the weak magnetism form factors in terms of the baryon magnetic moments.

XII. HYPERON NON-LEPTONIC DECAYS

Hyperon non-leptonic decays are $\Delta S = -1$ non-leptonic weak interaction processes[††]. The weak Hamiltonian is proportional to

$$(\bar{s}\gamma^\mu P_L u)\,(\bar{u}\gamma_\mu P_L d) \tag{12.1}$$

at the weak scale. At leading order in $1/N_c$, factorization is exact, so that the weak Hamiltonian can be written as the product of currents. As a result, large N_c considerations do not seem to lead to the $\Delta I = 1/2$ rule for K decays, one of the most striking features of non-leptonic weak interactions. It has been suggested that a naive application of $1/N_c$ counting is incorrect, however, because of large logarithms of the form $\ln M_{\rm W}/\Lambda_{\rm QCD}$ from renormalization group scaling (see refs. [22,23] for discussion of this issue). We do not consider the issue of whether the $\Delta I = 1/2$ rule can be derived in large N_c in this work. We will assume octet dominance and $\Delta I = 1/2$ enhancement in the following analysis of the hyperon non-leptonic decay amplitudes in the $1/N_c$ expansion. This assumption appears to be valid experimentally.

The general form of the decay amplitude for spin-1/2 baryons is [24]

$$\mathcal{M} = G_F m_{\pi^+}^2 \bar{u}_{B_f}\,(A - B\gamma_5)\,u_{B_i}, \tag{12.2}$$

where A and B are parity violating s-wave and parity conserving p-wave decay amplitudes. The decay rates and asymmetry parameters are given in terms of the amplitudes s and p which are related to A and B by

$$s = A, \quad p = B\frac{|\mathbf{p}_f|}{E_f + M_f}, \tag{12.3}$$

[††]Recall that S is defined to be strange quark number, not strangeness, in this work.

where $\mathbf{p}_f$, E_f and M_f are the three-momentum, energy and mass of the final baryon. The dimensionless amplitudes s are given in Table XVI, where we quote the experimental values extracted in ref. [25].

The assumption of octet dominance or $\Delta I = 1/2$ enhancement leads to three isospin relations amongst the seven decay amplitudes for the spin-1/2 octet baryons,

$$\begin{aligned}
&\mathcal{A}\left(\Sigma^+ \to n\pi^+\right) = \sqrt{2}\mathcal{A}\left(\Sigma^+ \to p\pi^0\right) + \mathcal{A}\left(\Sigma^- \to n\pi^-\right),\\
&\mathcal{A}\left(\Lambda^0 \to p\pi^-\right) = -\sqrt{2}\,\mathcal{A}\left(\Lambda^0 \to n\pi^0\right),\\
&\mathcal{A}\left(\Xi^- \to \Lambda^0\pi^-\right) = -\sqrt{2}\,\mathcal{A}\left(\Xi^0 \to \Lambda^0\pi^0\right),
\end{aligned} \tag{12.4}$$

where the relations eq. (12.4) apply to both the s- and p-wave amplitudes. There are two isospin relations amongst the five Ω^- p-wave amplitudes,

$$\begin{aligned}
&\mathcal{A}\left(\Omega^- \to \Xi^0\pi^-\right) = \sqrt{2}\,\mathcal{A}\left(\Omega^- \to \Xi^-\pi^0\right),\\
&\mathcal{A}\left(\Omega^- \to \Xi^{*0}\pi^-\right) = -\sqrt{2}\,\mathcal{A}\left(\Omega^- \to \Xi^{*-}\pi^0\right).
\end{aligned} \tag{12.5}$$

These isospin relations are evident in the experimental data.

A. S-Wave Decay Amplitudes

The s-wave hyperon non-leptonic decay amplitude does not vanish at zero momentum, and can be obtained using a soft pion theorem,

$$\mathcal{A}(B_i \to B_f + \pi^a) = \frac{i}{f_\pi}\left\langle B_f\right|\left[Q_5^a, \mathcal{H}_W\right]\left|B_i\right\rangle, \tag{12.6}$$

where $\mathcal{H}_W$ is the weak Hamiltonian, Q_5^a is the axial charge, and f_π is the pion decay constant. Since the weak Hamiltonian transforms as $(8,1)$ under chiral $SU(3)_L \times SU(3)_R$, $[Q_5^a, \mathcal{H}_W] = [Q^a, \mathcal{H}_W]$, where the vector charge $Q^a = I^a$. Thus, the s-wave non-leptonic weak decay amplitudes, which are obtained from matrix elements of $[Q^a, \mathcal{H}_W]$, involve the $1/N_c$ expansion for the weak Hamiltonian.

Assuming octet dominance, the weak Hamiltonian transforms as the $(6+i7)$ component of a $(0,8)$ representation of $SU(2) \times SU(3)$. The $1/N_c$ expansion for a $(0, adj)$ operator in the $SU(3)$ symmetry limit was derived in Section X. There are two operators series, given by the operators T^a and $\left\{J^i, G^{ia}\right\}/N_c$ times polynomials in J^2/N_c^2. Thus, the weak Hamiltonian has the expansion,

$$\mathcal{H}_W = b_1 T^{6+i7} + b_2 \frac{\left\{J^i, G^{i(6+i7)}\right\}}{N_c}, \tag{12.7}$$

up to corrections of relative order $1/N_c^2$. For baryons with strangeness of order N_c^0, T^{6+i7} is of order $\sqrt{N_c}$ and $\left\{J^i, G^{i(6+i7)}\right\}/N_c$ is of order $1/\sqrt{N_c}$, so the second term in the expansion eq. (12.7) is suppressed by a factor of $1/N_c$ relative to the first term. Thus, to leading order in the $1/N_c$ expansion, the s-wave non-leptonic decay amplitudes are given by the matrix elements of $[I^a, T^{6+i7}]$, and are purely F-coupling. At order $1/N_c$, the second term in eq. (12.7) produces a $1/N_c$-suppressed D-coupling. It is known that an $SU(3)$ symmetric fit works well for the s-wave amplitudes, with the D coupling smaller than the F coupling by about a factor of three [17].

At leading order in $1/N_c$, the s-wave amplitudes are pure F, and there are three relations amongst the four independent amplitudes,

$$\begin{aligned}
&\mathcal{A}\left(\Sigma^+ \to n\pi^+\right) = 0,\\
&\mathcal{A}\left(\Xi^- \to \Lambda^0\pi^-\right) = -\mathcal{A}\left(\Lambda^0 \to p\pi^-\right)\\
&\mathcal{A}\left(\Lambda^0 \to p\pi^-\right) = \sqrt{\frac{3}{2}}\mathcal{A}\left(\Sigma^- \to n\pi^-\right).
\end{aligned} \tag{12.8}$$

These three relations are valid up to a correction of relative order $1/N_c$. The one-parameter fit to the s-wave amplitudes is given in the third column of Table XVI. The fit agrees with the experimental data at the 30% level, which is consistent with the level expected for a $1/N_c$ correction.

When the subleading $1/N_c$ correction in eq. (12.7) is included, there are two relations valid to relative order $1/N_c^2$ amongst the four independent s-wave amplitudes. These relations are $\mathcal{A}\left(\Sigma^+ \to n\pi^+\right) = 0$, and the Lee-Sugawara relation,

$$\begin{aligned}
\sqrt{\frac{3}{2}}\,\mathcal{A}\left(\Sigma^- \to n\pi^-\right) + \mathcal{A}\left(\Lambda^0 \to p\pi^-\right)&\\
+2\,\mathcal{A}\left(\Xi^- \to \Lambda^0\pi^-\right) = 0.&
\end{aligned} \tag{12.9}$$

This two-parameter fit to the s-wave amplitudes is given in the fourth column of Table XVI. The fit agrees with the experimental data at the 10% level, which is the level expected for $1/N_c^2$ corrections.

The analysis of the s-wave amplitudes can be repeated adding perturbative $SU(3)$ symmetry breaking or using only $SU(2) \times U(1)$ flavor symmetry. The completely broken $SU(3)$ flavor symmetry case yields the same operator expansion as for perturbative symmetry breaking to linear order. The completely broken analysis is supplied here, since it gives results which are valid to all orders in $SU(3)$ symmetry breaking. The weak Hamiltonian for completely broken $SU(3)$ symmetry has the $1/N_c$ expansion,

$$\mathcal{H}_W = c_1 t_2^\dagger + c_2 \frac{\left\{J_{ud}^i, Y_2^{\dagger\, i}\right\}}{N_c} + c_3 \frac{\{N_s, t_2^\dagger\}}{N_c}, \tag{12.10}$$

up to corrections of relative order $1/N_c^2$. Note that $t_2^\dagger = T^{6+i7}$, so the leading operator in eq. (12.10) is the same as the leading operator in the $SU(3)$ symmetric expansion. At relative order $1/N_c$, the single operator in the $SU(3)$ symmetric expansion is replaced by two operators. These two operators are contained in $\{J^i, G^{i(6+i7)}\} = \{J_{ud}^i + J_s^i, Y_2^{\dagger\, i}\}$, since $\{J_s^i, Y_2^{\dagger\, i}\}$ reduces to a linear combination

of $t_2^\dagger$ and $\{N_s, t_2^\dagger\}$ by the operator identities. There are three parameters in the completely broken $SU(3)$ case, so there is one relation amongst the four independent s-wave amplitudes, $\mathcal{A}\left(\Sigma^+ \to n\pi^+\right) = 0$. This amplitude can be non-zero only due to corrections to the soft-pion limit.

Finally, note the subdominant $(0, 27)$ component of the weak Hamiltonian, which contains a $\Delta I = 3/2$ piece, is given by the two-body operator $\{t_1^\dagger, I^+\}$ at leading order in the $1/N_c$ expansion, so it is suppressed by one factor of $1/N_c$ relative to the leading $(0, 8)$ one-body operator $t_\alpha^\dagger$.

B. P-Wave Decay Amplitudes

The p-wave decay amplitude vanishes at zero-momentum, and so soft-pion theorems can not be used to simplify the calculation. The weak Hamiltonian transforms as $(0, 8)$, and the pion coupling transforms as part of an $SU(3)$ octet. Thus the p-wave hyperon non-leptonic amplitude transforms as $(1, 8\otimes8) = (1, 1)+(1, 8)+(1, 8)+(1, 10+\overline{10})+(1, 27)$. The operators contributing to the p-wave amplitudes were classified previously in the analysis of meson couplings with perturbative $SU(3)$ breaking in Sec. IX. The form of the $1/N_c$ expansion for the p-wave amplitudes in the $SU(3)$ symmetry limit is the same as the expansion for the meson couplings with perturbative $SU(3)$ symmetry breaking, except that the weak Hamiltonian transforms as T^{6+i7}, whereas the symmetry breaking Hamiltonian transforms as T^8. Thus, the expression derived in Sec. IX must be rotated to the $6+i7$ direction. The result for the p-wave amplitudes can be written as

$$\begin{aligned} P^{ia} = &\, d^{a\,(6+i7)\,c}\left(c_1 G^{ic} + c_2 \frac{J^i T^c}{N_c}\right) \\ &+ c_3 \frac{\{G^{ia}, T^{6+i7}\}}{N_c} + c_4 \frac{\{T^a, G^{i\,(6+i7)}\}}{N_c} \\ &+ \frac{c_5}{N_c}\left[J^2, \left[T^{6+i7}, G^{ia}\right]\right] \\ &+ c_6 \delta^{a\,(6+i7)} J^i, \end{aligned} \tag{12.11}$$

where a denotes the flavor of the pion (or kaon). The term proportional to c_6 does not contribute to any of the observed p-wave decay amplitudes. The double commutator term requires the initial and final baryons to have different spin, so it does not contribute to any of the octet baryon decay amplitudes, but does contribute to the p-wave Ω^- decays to octet baryons. The analysis of p-wave non-leptonic decay amplitudes in the chiral quark model [17] resembles the $1/N_c$ expansion (12.11). The chiral quark model also predicts the p-wave amplitudes in terms of five one-body and two-body operators.

There is a significant contribution to the p-wave decay amplitudes from pole graphs, which are sensitive to $SU(3)$ breaking. This introduces additional calculable operators to those given in eq. (12.11). The analysis of the p-wave amplitudes including pole graphs is complicated, and will be given elsewhere [12].

XIII. THE SKYRME REPRESENTATION

The large-N_c consistency conditions derived in ref. [1,2,3] can be analyzed by constructing irreducible representations of the contracted spin-flavor algebra using the theory of induced representations. This construction naturally leads to a description of large-N_c baryons in terms of the Skyrme representation[‡‡]. The analysis of large-N_c baryons using the Skyrme representation is discussed in detail in ref. [3], and will not be repeated here. In this section, we elucidate some connections between the quark and Skyrme representations of large-N_c baryons.

In the Skyrme representation, the space components of the axial currents are proportional to N_c times

$$X^{ia} = 2 \text{ Tr } A T^i A^{-1} T^a, \tag{13.1}$$

where A, the Skyrmion collective coordinate, is an element of $SU(F)$ in the F-flavor case. The spin operators J^i generate the right transformations of A,

$$A \to A U^{-1}, \tag{13.2}$$

where U is an element of a $SU(2)$ subgroup of $SU(F)$, and the flavor operators T^a generate left transformations

$$A \to U A, \tag{13.3}$$

where U is an element of $SU(F)$. The Skyrme representation gives an exact realization of the contracted spin-flavor algebra since X^{ia} is a c-number which satisfies the commutation relation $[X^{ia}, X^{jb}] = 0$ to all orders in the $1/N_c$ expansion.

The Skyrme and quark representations are equivalent in the large-N_c limit, but differ at subleading orders in the $1/N_c$ expansion. The equivalence is most transparent for the two-flavor case; for three or more flavors, there are additional subtleties.

The operator structure of the Skyrme representation is particularly simple for the case of two light flavors. The Skyrme representation operators J^i, I^a and X^{ia} have well-defined, N_c-independent matrix elements, whereas the quark representation operator G^{ia} has matrix elements of order N_c. The equivalence of the Skyrme and quark representations follows from the identification

$$X^{ia} = \lim_{N_c\to\infty}\ -\frac{4}{N_c+2}\, G^{ia} + \mathcal{O}\left(\frac{1}{N_c^2}\right). \tag{13.4}$$

[‡‡]We will refer to the Skyrme representation rather than the Skyrme model to emphasize that we are using the Skyrme model to provide an operator basis for the $1/N_c$ expansion of baryons in QCD, not in the Skyrme model.

The Skyrme representation identities derived in ref. [3] are reproduced by taking the limit eq. (13.4) of the quark representation identities in Table VII, and dropping subleading terms in $1/N_c$,

$$\begin{aligned}
X^{ia}X^{ia} &= 3,\\
X^{ia}J^i &= -I^a,\\
X^{ia}I^a &= -J^i,\\
\epsilon^{ijk}\epsilon^{abc}X^{ia}X^{jb} &= 2X^{ic},\\
I^2 &= J^2,\\
X^{ia}X^{ib} &= \delta^{ab},\\
\epsilon^{ijk}\left\{J^j, X^{ka}\right\} &= \epsilon^{abc}\left\{I^b, X^{ic}\right\},\\
X^{ia}X^{ja} &= \delta^{ij}.
\end{aligned} \tag{13.5}$$

Thus, the operator structure of the Skyrme and quark representations is identical at leading order in the $1/N_c$ expansion. Either operator basis can be used to expand QCD operators in a $1/N_c$ expansion. The two expansions will have different coefficients at subleading order in $1/N_c$, but will give identical predictions for physical quantities. The operator structure of the Skyrme representation is simpler than that of the quark representation for two flavors, since the Skyrme representation operators and operator identities do not depend on N_c. It is important to note, however, that the baryon spectrum in the Skyrme representation contains more states than in the quark representation. The baryon spectrum in the quark representation is a tower of states with $(J, I) = (1/2, 1/2),\ (3/2, 3/2), \ldots, (N_c/2, N_c/2)$. The spectrum in the Skyrme representation is an infinite tower of states $(J, I) = (1/2, 1/2),\ (3/2, 3/2), \ldots$. The extra states in the Skyrme model are sometimes regarded as "spurious" states from the point of view of the quark model. They have the quantum numbers of hadrons containing N_c quarks plus some $\bar q q$ pairs. The existence of extra states in the Skyrme representation does not affect the conclusion that the quark and Skyrme representations yield equivalent operator bases since any operator of finite spin (such as the mass operator with spin zero, or the axial current with spin one) does not couple states at the bottom of the tower to these additional states with spin of order N_c.

For two flavors, the quark representation operator G^{ia} can be written explicitly in terms of the Skyrme representation operator X^{ia} to all orders in $1/N_c$. The matrix elements of X^{ia} between baryon states are known [7]. The matrix elements of G^{ia} between baryon states can be computed using the method of ref. [26]. Baryons with $I_3 = 1/2$ are made of $(N_c+1)/2$ u-quarks combined into a state with spin J_u, and $(N_c-1)/2$ d-quarks combined into a state with spin J_d, where J_u and J_d are combined to form a state with total spin J. The matrix elements of J_u and J_d can be computed using standard methods [27] to obtain the matrix elements of G^{ia}. Writing the matrix elements of G^{ia} in terms of X^{ia}, one obtains

$$G^{ia} = -\frac{N_c+2}{4}\ \sqrt{1-\hat z}\ X^{ia} + \frac{1}{N_c+2}\left(\frac{1}{1+\sqrt{1-\hat z}}\right) J^i I^a, \tag{13.6}$$

where $\hat z$ is the operator

$$\hat z = \frac{2}{\left(N_c+2\right)^2} Ad_+ J^2, \tag{13.7}$$

and $Ad_+ J^2$ is the operator defined by

$$\left(Ad_+ J^2\right)\mathcal{O} \equiv \left\{J^2, \mathcal{O}\right\}. \tag{13.8}$$

Eq. (13.6) is valid for matrix elements of G^{ia} between states with $J \le N_c/2$. Matrix elements of G^{ia} in which at least one state has $J > N_c/2$ vanish.

The comparison of the Skyrme and quark representations is more interesting when the number of flavors is greater than two. We will concentrate on the three-flavor case in this discussion, since all the subtleties already occur for this case. The baryon spectrum of the Skyrme representation contains additional states which couple to baryon states with low spin. The baryon spectrum is determined by quantization of the collective coordinate A in eq. (13.1). The collective coordinate A must be quantized subject to the constraint that the body-centered hypercharge is $N_c/3$ (i.e. T^8 is $N_c/\sqrt{12}$) [20], as dictated by the Wess-Zumino term. It is important to note that this constraint depends on N_c. Many errors in the Skyrme model literature arise from quantizing the Skyrmion with the hypercharge set to its value for $N_c = 3$, $Y = 1$, while expanding in $1/N_c$. Because of this constraint, the spectrum of the Skyrme model depends explicitly on N_c, in contrast to the two-flavor case. The spectrum of the Skyrme representation contains the same tower of states as the quark representation. These representations, which are given in Table I, consist of Young tableaux with N_c boxes for spin and flavor. In addition, the Skyrme representation contains states which consist of Young tableaux with N_c+3 boxes, N_c+6 boxes, etc. [9]. These "spurious" states can have low spin, such as 1/2, 3/2, etc., and they can couple to the standard spin-1/2 and spin-3/2 baryons via operators of finite spin, such as the axial currents. These additional states have the quantum numbers of a state composed of N_c quarks and $\bar q q$ pairs. In large-N_c QCD, pair creation of an additional $\bar q q$ pair is suppressed by a factor of $1/\sqrt{N_c}$, since pair creation plus annihilation produces a closed fermion loop, which is down by $1/N_c$. It is straightforward to check by explicit computation of Clebsch-Gordan coefficients for arbitrary N_c that the matrix elements of operators between the "normal" and "spurious" states have this $1/N_c$ suppression. For example, the amplitude for a spin-1/2 baryon with N_c boxes to couple via the axial current to a baryon with N_c+3 boxes (i.e. a baryon with one extra $\bar q q$ pair) is suppressed by $1/\sqrt{N}$. Thus, the additional states of the Skyrme representation affect the couplings of the N_c-quark baryon states only at subleading order.

The Skyrme operators J^i, T^a and X^{ia} can be used to obtain a $1/N_c$ expansion for three flavors. Since the

matrix elements of the Skyrme operator X^{ia} now have a N_c-dependence, the relation eq. (13.4) between the quark and Skyrme operators G^{ia} and X^{ia} is no longer valid, and the Skyrme model operator identities are not given by taking the $N_c \to \infty$ limit of the quark model identities. The N_c-dependence of the matrix elements of the operators T^a and X^{ia} is different in different regions of the weight diagram. This non-trivial N_c-dependence of operator matrix elements is what made the analysis in ref. [3] of the $SU(3)$ flavor symmetry limit complicated. In ref. [3], the coupling of baryons to octet mesons was given in terms of two invariant amplitudes $\mathcal{M}$ and $\mathcal{N}$, with

$$\frac{\mathcal{N}}{\mathcal{M}} = \frac{1}{2} + \frac{\alpha}{N_c} + \mathcal{O}\left(\frac{1}{N_c^2}\right), \tag{13.9}$$

where α is an undetermined parameter. In the quark representation, one can show that the operator G^{ia} implies that

$$\frac{\mathcal{N}}{\mathcal{M}} = \frac{1}{2}\frac{N_c-1}{N_c+2} = \frac{1}{2} - \frac{3}{2N_c} + \ldots, \tag{13.10}$$

so that $\alpha = -3/2$ in the non-relativistic quark model. Similarly, the ratio $\mathcal{N}/\mathcal{M}$ can be calculated for arbitrary N_c for the Skyrme model operator X^{ia},

$$\frac{\mathcal{N}}{\mathcal{M}} = \frac{1}{2}\frac{(N_c-1)(N_c+9)}{N_c^2+8N_c+9} = \frac{1}{2} + \mathcal{O}\left(\frac{1}{N_c^2}\right), \tag{13.11}$$

so that $\alpha = 0$ in the Skyrme model. In the $1/N_c$ expansion of the meson couplings, the operator J^iT^a/N_c changes the prediction for the parameter α in either the quark or Skyrme representation away from these quark and Skyrme model values. The coefficient of J^iT^a/N_c must be different in the $1/N_c$ expansions in the quark and Skyrme representations to produce a given value of α since G^{ia} and X^{ia} give different contributions to α. Using eq. (9.11), one finds the non-trivial relation

$$X^{ia} = -\frac{4}{N_c+2}\left[G^{ia} - \frac{J^iT^a}{N_c}\right] + \mathcal{O}\left(\frac{1}{N_c^2}\right), \tag{13.12}$$

between the Skyrme and quark model operators,[§§] where the overall coefficient in front of the parentheses is determined by requiring that the Skyrme model identity $X^{ia}T^a = -J^i$ is satisfied to this order. Eq. (13.12) implies that some of the operator identities in the Skyrme and quark model are different, even at leading order in N_c. For instance, the Skyrme model identity $X^{ia}T^a = -J^i$ is valid in the case of two or three flavors, but the analogous quark model identity $\{T^a, G^{ia}\} = (N_c+F)(1-1/F)J^i$ has a different coefficient of proportionality for two and three flavors, even at leading order in $1/N_c$. This occurs because the J^iT^a/N_c term in eq. (13.12) for $a = 8$ is unsuppressed relative to G^{ia}, and it produces a leading order change in the $X^{ia}T^a$ identity. Using eq. (13.12) and eq. (5.12), one finds that

$$\begin{aligned} X^{ia}T^a &= -\frac{4}{N_c+2}\left[G^{ia}T^a - \frac{1}{N_c}J^iT^2\right] + \ldots \\ &= -\frac{4}{N_c+2}\left[\frac{1}{2}(N_c+F)\left(1-\frac{1}{F}\right)\right. \\ &\qquad \left. -\frac{1}{4F}(N_c+2F)(F-2)\right]J^i + \ldots \\ &= -J^i, \end{aligned} \tag{13.13}$$

for any number of flavors F.

We will not work out the operator identities in the Skyrme basis. Some of these identities are given in ref. [3].

§§The lefthand side of Eq. (13.12) is the Skyrme model X^{ia} projected onto the set of states which exist in the quark model. There are extra states in the Skyrme model. The full Skyrme model operator X^{ia} can not be written as an expansion in terms of quark operators since it has matrix elements between the quark model states and the extra Skyrme model states.

XIV. CONCLUSIONS

The $1/N_c$ expansion allows one to compute properties of baryons in a systematic expansion of QCD. There are only a few terms in the expansion at any given order in $1/N_c$ once redundant operators are eliminated using the operator reduction rule. Results which hold to two orders in $1/N_c$ typically work at the 10–15% level. For $N_c = 3$, the $1/N_c$ expansion usually can not be carried beyond second order because the number of independent operators becomes too large to make any non-trivial predictions. The $1/N_c$ expansion explains which features of the quark and Skyrme models follow directly from the spin-flavor symmetry structure of QCD.

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APPENDIX A: SU(Q) GROUP THEORY

This appendix reviews some $SU(Q)$ group theory which is needed for the derivation of the quark operator identities in Secs. IV and V. The results in this appendix can be applied to the spin-flavor group by setting $Q = 2F$, and to the flavor group by setting $Q = F$.

Irreducible representations of $SU(Q)$ are denoted in a number of different ways; these include their dimensions, Young tableaux, tensors, and Dynkin labels. The fundamental representation of $SU(Q)$ can be represented by its dimension Q, the Young tableau $\square$, the tensor with one upper index T^α, or the Dynkin label $[1,0,0,\ldots,0]$, where the number of entries in the Dynkin label is $(Q-1)$, the rank of $SU(Q)$. The complex conjugate representation of the fundamental also is dimension Q, and is often denoted by $\overline{Q}$. The Young tableau of $\overline{Q}$ is

$$\begin{array}{c}\square\\ \square\\ \square\\ \vdots\\ \square\\ \square\end{array} \tag{A1}$$

(sometimes written $\overline{\square}$), the completely antisymmetric product of $(Q-1)$ fundamental representations. $\overline{Q}$ also can be represented as the tensor with one lower index, T_β, or by the Dynkin label $[0,0,0,\ldots,0,1]$.

The tensor product of the fundamental representation and its complex conjugate contains the singlet and the adjoint representations of $SU(Q)$. The adjoint representation can be represented by its dimension (Q^2-1), the Young tableau

$$\begin{array}{ll}\square & \square\\ \square & \\ \square & \\ \vdots & \\ \square & \\ \square & \end{array} \quad , \tag{A2}$$

the traceless tensor T^α_β with one upper and one lower index, or the Dynkin label $[1,0,0,\ldots,0,1]$. The adjoint representation is a real representation. A tensor with one upper index α and one lower index β in the fundamental representation can be converted into a tensor with a single index A in the adjoint representation using the $SU(Q)$ group generators in the fundamental representation, Λ^A,

$$T^\alpha_\beta \to T^A = \left(\Lambda^A\right)^\beta_\alpha T^\alpha_\beta. \tag{A3}$$

The commutator of two $SU(Q)$ generators is defined by structure constants f^{ABC} which are completely antisymmetric and real,

$$\left[\Lambda^A, \Lambda^B\right] = i f^{ABC} \Lambda^C. \tag{A4}$$

The anticommutator of two $SU(Q)$ generators *in the fundamental representation* is

$$\left\{\Lambda^A, \Lambda^B\right\} = \frac{1}{Q}\delta^{AB} + d^{ABC}\Lambda^C, \tag{A5}$$

which defines the d-symbol, a real and completely symmetric tensor with three adjoint indices. The f- and d-symbols for $SU(Q)$ can be written in terms of traces of $SU(Q)$ generators in the fundamental representation,

$$\begin{aligned} f^{ABC} &= -2i \ \mathrm{Tr}\Lambda^A \left[\Lambda^B, \Lambda^C\right], \\ d^{ABC} &= 2 \ \mathrm{Tr}\Lambda^A \left\{\Lambda^B, \Lambda^C\right\}. \end{aligned} \tag{A6}$$

A number of identities for contractions of f- and d-symbols are used in the derivation of the quark operator identities. These are:

$$\begin{aligned} d^{AAB} &= 0, \\ d^{ABC}d^{ABD} &= \left(Q - \frac{4}{Q}\right) \delta^{CD}, \\ f^{ABC}f^{ABD} &= Q\ \delta^{CD}, \\ f^{ABC}f^{ADE}d^{BDF} &= \frac{Q}{2}\ d^{CEF}, \\ d^{ABC}d^{ADE}d^{BDF} &= \left(\frac{Q}{2} - \frac{6}{Q}\right) d^{CEF}, \\ d^{ABC}d^{ADE}f^{BDF} &= \left(\frac{Q}{2} - \frac{2}{Q}\right) f^{CEF}. \end{aligned} \tag{A7}$$

These identities can be proved using eq. (A6) and the trace identities

$$\begin{aligned} \mathrm{Tr}\ \Lambda^A X \Lambda^A Y &= \frac{1}{2}\mathrm{Tr}\ X \ \ \mathrm{Tr} Y - \frac{1}{2Q}\mathrm{Tr}\ XY, \\ \mathrm{Tr}\ \Lambda^A X \ \ \mathrm{Tr}\ \Lambda^A Y &= \frac{1}{2}\mathrm{Tr}\ XY - \frac{1}{2Q}\mathrm{Tr}\ X \mathrm{Tr}\ Y, \end{aligned} \tag{A8}$$

which follow from the Fierz identity

$$\left(\Lambda^A\right)^\alpha_\beta \left(\Lambda^A\right)^\gamma_\delta = \frac{1}{2}\ \delta^\alpha_\delta \delta^\gamma_\beta - \frac{1}{2Q}\ \delta^\alpha_\beta \delta^\gamma_\delta. \tag{A9}$$

The tensor product of two adjoint representations can be divided into the symmetric product $(adj \times adj)_S$ and the antisymmetric product $(adj \times adj)_A$. The irreducible representations in the symmetric and antisymmetric products of two adjoints are given in Tables III and XV, respectively. These decompositions can be written as

$$\begin{aligned} (adj \otimes adj)_S &= 1 + adj + \bar{a}a + \bar{s}s, \\ (adj \otimes adj)_A &= adj + \bar{a}s + \bar{s}a, \end{aligned} \tag{A10}$$

where the Dynkin labels of the irreducible representations in eq. (A10) are listed in Tables III and XV. The designation of the irreducible representations by adj, $\bar{a}a$, $\bar{s}s$, $\bar{a}s$ and $\bar{s}a$ is used throughout this paper. This designation is not conventional. The dimensions, Casimirs, Dynkin labels and Young tableaux for these representations are listed in Table XIV.

The decomposition of the tensor product of two adjoints into the irreducible representations in eq. (A10) is straightforward using tensor methods, if the adjoint representation is given as a tensor T^α_β with indices in the fundamental representation. The derivation of the quark operator identities requires the decomposition when the adjoint representation is given as a tensor T^A with an index in the adjoint representation.

Consider first the decomposition of $(adj \times adj)_A$. Given two adjoints T_1^A and T_2^A, one can define the antisymmetric tensor

$$\mathcal{X}_-^{AB} = T_1^A T_2^B - T_1^B T_2^A. \tag{A11}$$

The tensor $\mathcal{X}_-^{AB}$, which is antisymmetric in the two adjoint indices, can be decomposed into $adj + \bar{a}s + \bar{s}a$. The adjoint is obtained by contracting with the f-symbol,

$$\mathcal{X}^C_{-,adj} = f^{ABC} \mathcal{X}_-^{AB}. \tag{A12}$$

The $\bar{a}s + \bar{s}a$ representations can be obtained by subtracting the adjoint from $\mathcal{X}_-^{AB}$

$$\mathcal{X}^{AB}_{\bar{a}s+\bar{s}a} = \mathcal{X}_-^{AB} - \frac{1}{Q} f^{ABC} f^{CGH} \mathcal{X}_-^{GH}. \tag{A13}$$

The coefficient of the last term has been chosen so that

$$f^{ABC}\ \mathcal{X}^{AB}_{\bar{a}s+\bar{s}a} = 0. \tag{A14}$$

The adjoint representation is real, so $\bar{a}s$ and $\bar{s}a$ are conjugate representations.

The decomposition of the symmetric product of two adjoints is more involved. Define the symmetric tensor

$$\mathcal{X}_+^{AB} = T_1^A T_2^B + T_1^B T_2^A. \tag{A15}$$

The tensor $\mathcal{X}_+^{AB}$, which is symmetric in the two adjoint indices, can be decomposed into $1 + adj + \bar{a}a + \bar{s}s$. The singlet is obtained by contracting the two adjoint indices, whereas the adjoint representation in $\mathcal{X}_+^{AB}$ can be projected out using the d-symbol,

$$\begin{aligned} \mathcal{X}_1 &= \mathcal{X}_+^{AA}, \\ \mathcal{X}^C_{+,adj} &= d^{ABC} \mathcal{X}_+^{AB}. \end{aligned} \tag{A16}$$

The sum of the $\bar{a}a$ and $\bar{s}s$ representations is given by subtracting off the singlet and adjoint components of $\mathcal{X}_+^{AB}$,

$$\begin{aligned} \mathcal{X}^{AB}_{\bar{a}a+\bar{s}s} = \mathcal{X}_+^{AB} &- \frac{1}{Q^2-1} \delta^{AB}\ \mathcal{X}_+^{CC} \\ &- \frac{Q}{Q^2-4} d^{ABC} d^{CGH}\ \mathcal{X}_+^{GH}. \end{aligned} \tag{A17}$$

Eq. (A17) can be separated into the individual $\bar{a}a$ and $\bar{s}s$ representations using a trick. Any tensor X^{AB} with two adjoint indices transforms under infinitesimal $SU(Q)$ transformations by the generators in the adjoint representation (which are the structure constants) acting on the two indices separately

$$\delta^C \mathcal{X}^{AB} = -i f^{CAD} \mathcal{X}^{DB} - i f^{CBD} \mathcal{X}^{AD}. \tag{A18}$$

Thus, the Casimir operator C_2 acting on $\mathcal{X}^{AB}$ is

$$\begin{aligned} (C_2 \mathcal{X})^{AB} = -\Big(& f^{CAE} f^{CEG} \delta^{BH} + f^{CBE} f^{CEH} \delta^{AG} \\ & + 2 f^{CAG} f^{CBH} \Big) \mathcal{X}^{GH}. \end{aligned} \tag{A19}$$

The Casimir operator has the values (see Table XIV)

$$C_2 = \begin{cases} 2Q & \bar{a}s, \\ 2Q & \bar{s}a\ , \\ 2(Q-1) & \bar{a}a\ , \\ 2(Q+1) & \bar{s}s\ , \end{cases} \tag{A20}$$

in the representations of interest. Using eqs. (A7,A20), we find that

$$f^{ACG} f^{BCH} \mathcal{X}^{GH} = \begin{cases} 0 & \bar{a}s\ , \\ 0 & \bar{s}a\ , \\ \mathcal{X}^{AB} & \bar{a}a\ , \\ -\mathcal{X}^{AB} & \bar{s}s\ . \end{cases} \tag{A21}$$

The operator $f^{ACG} f^{BCH}$ can be used to split $\mathcal{X}^{AB}_{\bar{a}a+\bar{s}s}$ into $\bar{a}a$ and $\bar{s}s$ representations, using the projection operators

$$\begin{aligned} (P_{\bar{a}a}\mathcal{X})^{AB} &= \frac{1}{2} \left(\delta^{AG}\delta^{BH} + f^{ACG} f^{BCH} \right) \mathcal{X}^{GH}, \\ (P_{\bar{s}s}\mathcal{X})^{AB} &= \frac{1}{2} \left(\delta^{AG}\delta^{BH} - f^{ACG} f^{BCH} \right) \mathcal{X}^{GH}. \end{aligned} \tag{A22}$$

Other useful identities are

$$d^{ACG} d^{BCH} \mathcal{X}^{GH} = \begin{cases} 0 & \bar{a}s\ , \\ 0 & \bar{s}a\ , \\ -\left(1 + 2/Q\right) \mathcal{X}^{AB} & \bar{a}a\ , \\ \left(1 - 2/Q\right) \mathcal{X}^{AB} & \bar{s}s\ , \end{cases} \tag{A23}$$

$$f^{ACG} d^{BCH} \mathcal{X}^{GH} = \begin{cases} -i\mathcal{X}^{AB} & \bar{a}s\ , \\ i\mathcal{X}^{AB} & \bar{s}a\ , \\ -i\mathcal{X}^{AB} & \bar{a}a\ , \\ i\mathcal{X}^{AB} & \bar{s}s\ , \end{cases} \tag{A24}$$

so that $f^{ACG} d^{BCH}$ can be used to split $\mathcal{X}^{AB}_{\bar{a}s+\bar{s}a}$ into $\bar{a}s$ and $\bar{s}a$ representations.

APPENDIX B: SU(2F) → SU(2) × SU(F)

To decompose the two-body quark operator identities into irreducible representations of $SU(2) \times SU(F)$, we need the decomposition of the $SU(2F)$ irreducible representations adj, $\bar{a}a$ and $\bar{s}s$ into $SU(2) \times SU(F)$ representations. A straightforward computation yields

$$\begin{aligned} adj &\rightarrow (0, adj) + (1, 0) + (1, adj), \\ \bar{a}a &\rightarrow (0,0) + (0, adj) + (0, \bar{a}a) + (0, \bar{s}s) \\ &\quad + (1, adj) + (1, adj) + (1, \bar{a}a) + (1, \bar{a}s + \bar{s}a) \\ &\quad + (2,0) + (2, adj) + (2, \bar{a}a), \\ \bar{s}s &\rightarrow (0,0) + (0, adj) + (0, \bar{a}a) + (0, \bar{s}s) \\ &\quad + (1, adj) + (1, adj) + (1, \bar{s}s) + (1, \bar{a}s + \bar{s}a) \\ &\quad + (2,0) + (2, adj) + (2, \bar{s}s). \end{aligned} \tag{B1}$$

For the special cases of $SU(4) \rightarrow SU(2) \times SU(2)$ and $SU(6) \rightarrow SU(2) \times SU(3)$, eq. (B1) reduces to

$$
\begin{aligned}
&15 \rightarrow (0,1)+(1,0)+(1,1),\\
&20 \rightarrow (0,0)+(1,1)+(2,0)+(0,2),\\
&84 \rightarrow (0,0)+(1,1)+(1,1)+(0,2)+(2,0)\\
&\quad +(1,2)+(2,1)+(2,2),
\end{aligned} \tag{B2}
$$

and

$$
\begin{aligned}
&35 \rightarrow (0,8)+(1,0)+(1,8),\\
&189 \rightarrow (0,0)+(0,8)+(0,27)+(1,8)+(1,8)\\
&\qquad +(1,10+\overline{10})+(2,0)+(2,8),\\
&405 \rightarrow (0,0)+(0,8)+(0,27)+(1,8)+(1,8)\\
&\quad +(1,27)+(1,10+\overline{10})+(2,0)+(2,8)+(2,27),
\end{aligned} \tag{B3}
$$

respectively.

The decomposition of the $SU(2F)$ d-symbol d^{ABC} for $SU(2F) \rightarrow SU(2) \times SU(F)$ is required. The adjoint of $SU(2F)$ decomposes into the representations $(1,0)$, $(0,adj)$ and $(1,adj)$ of $SU(2) \times SU(F)$, so an adjoint index A is replaced by the $SU(2) \times SU(F)$ indices i, a and ia, respectively. The $SU(2F)$ d-symbol can be written as

$$
d^{ABC} = \begin{cases}
0 & \{A,B,C\} = \{i,j,k\}\\
0 & \{A,B,C\} = \{i,j,a\}\\
0 & \{A,B,C\} = \{i,a,b\}\\
\frac{1}{\sqrt{2}} d^{abc} & \{A,B,C\} = \{a,b,c\}\\
0 & \{A,B,C\} = \{ia,j,k\}\\
\frac{1}{\sqrt{F}} \delta^{ij}\delta^{ab} & \{A,B,C\} = \{ia,j,b\}\\
0 & \{A,B,C\} = \{ia,b,c\}\\
0 & \{A,B,C\} = \{ia,jb,k\}\\
\frac{1}{\sqrt{2}} \delta^{ij} d^{abc} & \{A,B,C\} = \{ia,jb,c\}\\
-\frac{1}{\sqrt{2}} \epsilon^{ijk} f^{abc} & \{A,B,C\} = \{ia,jb,kc\}
\end{cases} \tag{B4}
$$

in terms of the $SU(F)$ d- and f-symbols.

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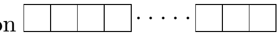
TABLE I. $SU(2) \otimes SU(F)$ decomposition of the $SU(2F)$ representation  of the ground state baryons. All Young tableaux contain N_c boxes.

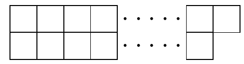
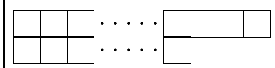
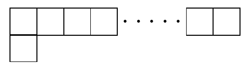
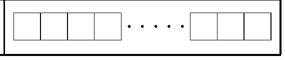
$SU(2)$	$SU(F)$
$\frac{1}{2}$	
$\frac{3}{2}$	
$\vdots$	$\vdots$
$\frac{N_c-2}{2}$	
$\frac{N_c}{2}$	

TABLE II. $SU(2F)$ Commutation Relations

$$\left[J^i, T^a\right] = 0,$$
$$\left[J^i, J^j\right] = i\epsilon^{ijk} J^k, \qquad \left[T^a, T^b\right] = i f^{abc} T^c,$$
$$\left[J^i, G^{ja}\right] = i\epsilon^{ijk} G^{ka}, \qquad \left[T^a, G^{ib}\right] = i f^{abc} G^{ic},$$
$$\left[G^{ia}, G^{jb}\right] = \tfrac{i}{4}\delta^{ij} f^{abc} T^c + \tfrac{i}{2F}\delta^{ab}\epsilon^{ijk} J^k + \tfrac{i}{2}\epsilon^{ijk} d^{abc} G^{kc}.$$

TABLE III. $(adj \otimes adj)_S$

	$SU(Q)$	$SU(6)$	$SU(4)$
$(adj \otimes adj)_S$	$\left([1,0,0,0,\ldots,0,0,1]^2 \right)_S$	$\left([1,0,0,0,1]^2 \right)_S$	$\left([1,0,1]^2 \right)_S$
1	$[0,0,0,0,\ldots,0,0,0]$	$[0,0,0,0,0]$	$[0,0,0]$
adj	$[1,0,0,0,\ldots,0,0,1]$	$[1,0,0,0,1]$	$[1,0,1]$
$\bar{a}a$	$[0,1,0,0,\ldots,0,1,0]$	$[0,1,0,1,0]$	$[0,2,0]$
$\bar{s}s$	$[2,0,0,0,\ldots,0,0,2]$	$[2,0,0,0,2]$	$[2,0,2]$

The $SU(6)$ and $SU(4)$ representations are 1+35+189+405 and 1+15+20+84, respectively.

TABLE IV. $(adj \otimes adj \otimes adj)_S$

$SU(Q)$	$SU(6)$	$SU(4)$
$\Big([1,0,0,0,\ldots,0,0,1]^3 \Big)_S$	$\Big([1,0,0,0,1]^3 \Big)_S$	$\Big([1,0,1]^3 \Big)_S$
$[0,0,0,0,\ldots,0,0,0]$	$[0,0,0,0,0]$	$[0,0,0]$
$[1,0,0,0,\ldots,0,0,1]$	$[1,0,0,0,1]$	$[1,0,1]$
$[1,0,0,0,\ldots,0,0,1]$	$[1,0,0,0,1]$	$[1,0,1]$
$[0,0,1,0,\ldots,1,0,0]$	$[0,0,2,0,0]$	
$[0,1,0,0,\ldots,0,1,0]$	$[0,1,0,1,0]$	
$[2,0,0,0,\ldots,0,1,0]$	$[2,0,0,1,0]$	$[2,1,0]$
$[0,1,0,0,\ldots,0,0,2]$	$[0,1,0,0,2]$	$[0,1,2]$
$[2,0,0,0,\ldots,0,0,2]$	$[2,0,0,0,2]$	$[2,0,2]$
$[1,1,0,0,\ldots,0,1,1]$	$[1,1,0,1,1]$	$[1,2,1]$
$[3,0,0,0,\ldots,0,0,3]$	$[3,0,0,0,3]$	$[3,0,3]$

TABLE V. $(\bar{a}a \otimes adj)$

$SU(Q)$	$SU(6)$	$SU(4)$
$([0,1,0,0,\ldots,0,1,0] \otimes adj)$	$([0,1,0,1,0] \otimes adj)$	$([0,2,0] \otimes adj)$
$[1,0,0,0,\ldots,0,0,1]$	$[1,0,0,0,1]$	$[1,0,1]$
$[0,1,0,0,\ldots,0,1,0]$	$[0,1,0,1,0]$	
$[0,1,0,0,\ldots,0,1,0]$	$[0,1,0,1,0]$	
$[0,0,1,0,\ldots,1,0,0]$	$[0,0,2,0,0]$	$[0,2,0]$
$[2,0,0,0,\ldots,0,1,0]$	$[2,0,0,1,0]$	$[2,1,0]$
$[0,1,0,0,\ldots,0,0,2]$	$[0,1,0,0,2]$	$[0,1,2]$
$[1,1,0,0,\ldots,1,0,0]$	$[1,1,1,0,0]$	
$[0,0,1,0,\ldots,0,1,1]$	$[0,0,1,1,1]$	
$[1,1,0,0,\ldots,0,1,1]$	$[1,1,0,1,1]$	$[1,2,1]$

TABLE VI. $SU(2F)$ Identities: The second column gives the transformation properties of the identities under $SU(2)\times SU(F)$.

$2\ \{J^i,J^i\}+F\ \{T^a,T^a\}+4F\ \{G^{ia},G^{ia}\}=N\,(N+2F)\,(2F-1)$	$(0,0)$
$d^{abc}\ \{G^{ia},G^{ib}\}+\frac{2}{F}\ \{J^i,G^{ic}\}+\frac{1}{4}\ d^{abc}\ \{T^a,T^b\}=(N+F)\left(1-\frac{1}{F}\right)\ T^c$	$(0,adj)$
$\{T^a,G^{ia}\}=(N+F)\left(1-\frac{1}{F}\right)\ J^i$	$(1,0)$
$\frac{1}{F}\ \{J^k,T^c\}+d^{abc}\ \{T^a,G^{kb}\}-\epsilon^{ijk}f^{abc}\{G^{ia},G^{jb}\}=2\,(N+F)\left(1-\frac{1}{F}\right)\ G^{kc}$	$(1,adj)$
$4F\,(2-F)\ \{G^{ia},G^{ia}\}+3F^2\ \{T^a,T^a\}+4\left(1-F^2\right)\ \{J^i,J^i\}=0$	$(0,0)$
$(4-F)\,d^{abc}\ \{G^{ia},G^{ib}\}+\frac{3}{4}F\ d^{abc}\ \{T^a,T^b\}-2\left(F-\frac{4}{F}\right)\ \{J^i,G^{ia}\}=0$	$(0,adj)$
$4\ \{G^{ia},G^{ib}\}=-3\ \{T^a,T^b\}\qquad(\bar{a}a)$	$(0,\bar{a}a)$
$4\ \{G^{ia},G^{ib}\}=\{T^a,T^b\}\qquad(\bar{s}s)$	$(0,\bar{s}s)$
$\epsilon^{ijk}\ \{J^i,G^{jc}\}=f^{abc}\ \{T^a,G^{kb}\}$	$(1,adj)$
$d^{abc}\ \{T^a,G^{kb}\}=\left(1-\frac{2}{F}\right)\left(\{J^k,T^c\}-\epsilon^{ijk}f^{abc}\ \{G^{ia},G^{jb}\}\right)$	$(1,adj)$
$\epsilon^{ijk}\ \{G^{ia},G^{jb}\}=f^{acg}d^{bch}\ \{T^g,G^{kh}\}\qquad(\bar{a}s+\bar{s}a)$	$(1,\bar{a}s+\bar{s}a)$
$\{T^a,G^{ib}\}=0\qquad(\bar{a}a)$	$(1,\bar{a}a)$
$\{G^{ia},G^{ja}\}=\frac{1}{2}\left(1-\frac{1}{F}\right)\ \{J^i,J^j\}\qquad(J=2)$	$(2,0)$
$d^{abc}\ \{G^{ia},G^{jb}\}=\left(1-\frac{2}{F}\right)\ \{J^i,G^{jc}\}\qquad(J=2)$	$(2,adj)$
$\{G^{ia},G^{jb}\}=0\qquad(J=2,\bar{a}a)$	$(2,\bar{a}a)$

TABLE VII. $SU(4)$ Identities: The second column gives the transformation properties of the identities under $SU(2)\times SU(2)$.

$\{J^i,J^i\}+\{I^a,I^a\}+4\ \{G^{ia},G^{ia}\}=\frac{3}{2}N\,(N+4)$	$(0,0)$
$2\,\{J^i,G^{ia}\}=(N+2)\ I^a$	$(0,1)$
$2\,\{I^a,G^{ia}\}=(N+2)\ J^i$	$(1,0)$
$\frac{1}{2}\ \{J^k,I^c\}-\epsilon^{ijk}\epsilon^{abc}\{G^{ia},G^{jb}\}=(N+2)\ G^{kc}$	$(1,1)$
$\{I^a,I^a\}-\{J^i,J^i\}=0$	$(0,0)$
$4\ \{G^{ia},G^{ib}\}=\{I^a,I^b\}\qquad(I=2)$	$(0,2)$
$\epsilon^{ijk}\ \{J^i,G^{jc}\}=\epsilon^{abc}\ \{I^a,G^{kb}\}$	$(1,1)$
$4\ \{G^{ia},G^{ja}\}=\{J^i,J^j\}\qquad(J=2)$	$(2,0)$

TABLE VIII. $SU(6)$ Identities: The second column gives the transformation properties of the identities under $SU(2) \times SU(3)$.

$2\ \{J^i, J^i\} + 3\ \{T^a, T^a\} + 12\ \{G^{ia}, G^{ia}\} = 5N\,(N+6)$	$(0,0)$
$d^{abc}\ \{G^{ia}, G^{ib}\} + \frac{2}{3}\ \{J^i, G^{ic}\} + \frac{1}{4}\ d^{abc}\ \{T^a, T^b\} = \frac{2}{3}\,(N+3)\ T^c$	$(0, adj)$
$\{T^a, G^{ia}\} = \frac{2}{3}\,(N+3)\ J^i$	$(1,0)$
$\frac{1}{3}\ \{J^k, T^c\} + d^{abc}\ \{T^a, G^{kb}\} - \epsilon^{ijk} f^{abc} \{G^{ia}, G^{jb}\} = \frac{4}{3}\,(N+3)\ G^{kc}$	$(1, adj)$
$-12\ \{G^{ia}, G^{ia}\} + 27\ \{T^a, T^a\} - 32\ \{J^i, J^i\} = 0$	$(0,0)$
$d^{abc}\ \{G^{ia}, G^{ib}\} + \frac{9}{4}\ d^{abc}\ \{T^a, T^b\} - \frac{10}{3}\ \{J^i, G^{ic}\} = 0$	$(0, adj)$
$4\ \{G^{ia}, G^{ib}\} = \{T^a, T^b\} \qquad (\bar{s}s)$	$(0, \bar{s}s)$
$\epsilon^{ijk}\ \{J^i, G^{jc}\} = f^{abc}\ \{T^a, G^{kb}\}$	$(1, adj)$
$3\ d^{abc}\ \{T^a, G^{kb}\} = \{J^k, T^c\} - \epsilon^{ijk} f^{abc}\ \{G^{ia}, G^{jb}\}$	$(1, adj)$
$\epsilon^{ijk}\ \{G^{ia}, G^{jb}\} = f^{acg} d^{bch}\ \{T^g, G^{kh}\} \qquad (\bar{a}s + \bar{s}a)$	$(1, \bar{a}s + \bar{s}a)$
$3\ \{G^{ia}, G^{ja}\} = \{J^i, J^j\} \qquad (J=2)$	$(2,0)$
$3\ d^{abc}\ \{G^{ia}, G^{jb}\} = \{J^i, G^{jc}\} \qquad (J=2)$	$(2, adj)$

TABLE IX. Operator reduction for F flavors. The second column gives the allowed $SU(2) \times SU(F)$ representations for the operators in column one. The third column gives the combinations left after eliminating all redundant linear combinations using the identities in Table VI.

$\{J^i, J^j\}$	$(0,0)\ (2,0)$	$(0,0)\ (2,0)$
$\{G^{ia}, G^{jb}\}$	$(0,0)\ (0,adj)\ (0,\bar{a}a)\ (0,\bar{s}s)\ (1,adj)$ $(1,\bar{a}s+\bar{s}a)\ (2,0)\ (2,adj)\ (2,\bar{a}a)\ (2,\bar{s}s)$	$(2,\bar{s}s)$
$\{T^a, T^b\}$	$(0,0)\ (0,adj)\ (0,\bar{a}a)\ (0,\bar{s}s)$	$(0,\bar{a}a)\ (0,\bar{s}s)$
$\{J^i, T^a\}$	$(1,adj)$	$(1,adj)$
$\{J^i, G^{ja}\}$	$(0,adj)\ (1,adj)\ (2,adj)$	$(0,adj)\ (1,adj)\ (2,adj)$
$\{T^a, G^{ib}\}$	$(1,0)\ (1,adj)\ (1,adj)\ (1,\bar{a}s+\bar{s}a)$ $(1,\bar{a}a)\ (1,\bar{s}s)$	$(1,\bar{a}s+\bar{s}a)$ $(1,\bar{s}s)$

TABLE X. $SU(4)\times SU(2)\times U(1)$ Commutation Relations

$$\left[J^i_{ud},J^j_s\right]=0,\qquad \left[J^i_{ud},I^a\right]=0,\qquad \left[J^i_s,I^a\right]=0,$$

$$\left[J^i_s,G^{ja}\right]=0,\qquad \left[Y^{i\alpha},Y^{j\beta}\right]=0,\qquad \left[t^\alpha,t^\beta\right]=0,$$

$$\left[J^i_{ud},J^j_{ud}\right]=i\epsilon^{ijk}J^k_{ud},\qquad \left[J^i_s,J^j_s\right]=i\epsilon^{ijk}J^k_s,\qquad \left[I^a,I^b\right]=i\epsilon^{abc}I^c,$$

$$\left[J^i_{ud},G^{ja}\right]=i\epsilon^{ijk}G^{ka},\qquad \left[I^a,G^{ib}\right]=i\epsilon^{abc}G^{ic},\qquad \left[J^i_s,t^\alpha\right]=Y^{i\alpha},$$

$$\left[J^i_{ud},t^\alpha\right]=-Y^{i\alpha},\qquad \left[Y^{i\alpha},t^\beta\right]=0,\qquad \left[I^a,t^\alpha\right]=-\left(\frac{\tau^a}{2}\right)^\alpha_\beta t^\beta,$$

$$\left[N_s,I^a\right]=0,\qquad \left[N_s,J^i_{ud}\right]=0,\qquad \left[N_s,J^i_s\right]=0,$$

$$\left[N_s,G^{ia}\right]=0,\qquad \left[N_s,t^\alpha\right]=t^\alpha,\qquad \left[N_s,Y^{i\alpha}\right]=Y^{i\alpha},$$

$$\left[G^{ia},t^\alpha\right]=-\left(\frac{\tau^a}{2}\right)^\alpha_\beta Y^{i\beta},\qquad \left[I^a,Y^{i\alpha}\right]=-\left(\frac{\tau^a}{2}\right)^\alpha_\beta Y^{i\beta},$$

$$\left[J^i_{ud},Y^{j\alpha}\right]=\frac{i}{2}\epsilon^{ijk}Y^{k\alpha}-\frac{1}{4}\delta^{ij}t^\alpha,\qquad \left[G^{ia},G^{jb}\right]=\frac{i}{4}\delta^{ij}\epsilon^{abc}I^c+\frac{i}{4}\delta^{ab}\epsilon^{ijk}J^k_{ud},$$

$$\left[J^i_s,Y^{j\alpha}\right]=\frac{i}{2}\epsilon^{ijk}Y^{k\alpha}+\frac{1}{4}\delta^{ij}t^\alpha,\qquad \left[t^\alpha,t^\dagger_\beta\right]=\frac{1}{2}\delta^\alpha_\beta\left(3N_s-N\right)-2\left(\frac{\tau^a}{2}\right)^\alpha_\beta I^a,$$

$$\left[Y^{i\alpha},t^\dagger_\beta\right]=\delta^\alpha_\beta\left(J^i_s-\frac{1}{2}J^i_{ud}\right)-2\left(\frac{\tau^a}{2}\right)^\alpha_\beta G^{ia},\qquad \left[G^{ia},Y^{j\alpha}\right]=-\left(\frac{\tau^a}{2}\right)^\alpha_\beta\left(\frac{1}{4}\delta^{ij}t^\beta-\frac{i}{2}\epsilon^{ijk}Y^{k\beta}\right),$$

$$\left[Y^{i\alpha},Y^{\dagger j}_\beta\right]=\frac{1}{8}\delta^\alpha_\beta\delta^{ij}\left(3N_s-N\right)+\frac{i}{2}\delta^\alpha_\beta\epsilon^{ijk}\left(J^k_s+\frac{1}{2}J^k_{ud}\right)+\left(\frac{\tau^a}{2}\right)^\alpha_\beta\left(-\frac{1}{2}\delta^{ij}I^a+i\epsilon^{ijk}G^{ka}\right),$$

TABLE XI. $SU(4) \times SU(2) \times U(1)$ Identities for $\Delta S = 0$. The second column gives the $(J, I)_S$ quantum numbers of the operator identities.

$\left\{G^{ia}, G^{ia}\right\} = -\frac{1}{2}\left\{J^i_{ud}, J^i_{ud}\right\} + \frac{3}{8}\left(N - N_s\right)\left(N - N_s + 4\right)$	$(0,0)_0$
$\left\{J^i_s, J^i_s\right\} = \frac{1}{2} N_s \left(N_s + 2\right)$	$(0,0)_0$
$\left\{I^a, I^a\right\} = \left\{J^i_{ud}, J^i_{ud}\right\}$	$(0,0)_0$
$\left\{t^\alpha, t^\dagger_\alpha\right\} = 2 \ \left\{J^i_{ud}, J^i_s\right\} + N + N_s + N N_s - N_s^2$	$(0,0)_0$
$\left\{Y^{i\alpha}, Y^{\dagger i}_\alpha\right\} = -\frac{1}{2} \ \left\{J^i_{ud}, J^i_s\right\} + \frac{3}{4}\left(N + N_s + N N_s - N_s^2\right)$	$(0,0)_0$
$\left\{G^{ia}, J^i_{ud}\right\} = \frac{1}{2}\left(N - N_s + 2\right) \ I^a$	$(0,1)_0$
$\left(\frac{\tau^a}{2}\right)^\alpha_\beta \ \left\{t^\beta, t^\dagger_\alpha\right\} = \left(N_s + 1\right) I^a + 2 \ \left\{G^{ia}, J^i_s\right\}$	$(0,1)_0$
$\left\{Y^{i\beta}, Y^{\dagger i}_\alpha\right\} = \frac{3}{4} \ \left(N_s + 1\right) I^a - \frac{1}{2} \ \left\{G^{ia}, J^i_s\right\}$	$(0,1)_0$
$\left\{G^{ia}, G^{ib}\right\} = \frac{1}{4} \ \left\{I^a, I^b\right\} \qquad (I = 2)$	$(0,2)_0$
$\left\{G^{ia}, I^a\right\} = \frac{1}{2}\left(N - N_s + 2\right) \ J^i_{ud}$	$(1,0)_0$
$\left\{Y^{k\alpha}, t^\dagger_\alpha\right\} = i\epsilon^{ijk} \ \left\{J^i_{ud}, J^j_s\right\} + \left(N_s + 1\right) J^i_{ud} + \left(N - N_s + 2\right) J^i_s$	$(1,0)_0$
$\left\{Y^{\dagger k}_\alpha, t^\alpha\right\} = -i\epsilon^{ijk} \ \left\{J^i_{ud}, J^j_s\right\} + \left(N_s + 1\right) J^i_{ud} + \left(N - N_s + 2\right) J^i_s$	$(1,0)_0$
$i\epsilon^{ijk} \ \left\{Y^{i\alpha}, Y^{\dagger j}_\alpha\right\} = \left(N_s + 1\right) J^k_{ud} - \left(N - N_s + 2\right) J^k_s$	$(1,0)_0$
$i\epsilon^{ijk} \ \left\{G^{ic}, J^j_{ud}\right\} = i\epsilon^{abc}\left\{G^{ka}, I^b\right\}$	$(1,1)_0$
$\epsilon^{ijk}\epsilon^{abc} \ \left\{G^{ia}, G^{jb}\right\} = \frac{1}{2} \ \left\{J^k_{ud}, I^c\right\} - \left(N - N_s + 2\right) G^{kc}$	$(1,1)_0$
$\left(\frac{\tau^a}{2}\right)^\alpha_\beta \left\{Y^{k\beta}, t^\dagger_\alpha\right\} = i\epsilon^{ijk}\left\{G^{ia}, J^j_s\right\} + \frac{1}{2} \ \left\{J^i_s, I^a\right\} + \left(N_s + 1\right) G^{ia}$	$(1,1)_0$
$\left(\frac{\tau^a}{2}\right)^\alpha_\beta \left\{Y^{\dagger i}_\alpha, t^\beta\right\} = -i\epsilon^{ijk}\left\{G^{ia}, J^j_s\right\} + \frac{1}{2} \ \left\{J^i_s, I^a\right\} + \left(N_s + 1\right) G^{ia}$	$(1,1)_0$
$i\epsilon^{ijk}\left(\frac{\tau^a}{2}\right)^\alpha_\beta \left\{Y^{i\beta}, Y^{\dagger j}_\alpha\right\} = -\frac{1}{2} \ \left\{J^k_s, I^a\right\} + \left(N_s + 1\right) G^{ka}$	$(1,1)_0$
$\left\{G^{ia}, G^{ja}\right\} = \frac{1}{4} \ \left\{J^i_{ud}, J^j_{ud}\right\} \qquad (J = 2)$	$(2,0)_0$
$\left\{Y^{i\alpha}, Y^{\dagger j}_\alpha\right\} = \left\{J^i_{ud}, J^j_s\right\} \qquad (J = 2)$	$(2,0)_0$
$\left\{G^{ia}, J^j_s\right\} = \left(\frac{\tau^a}{2}\right)^\alpha_\beta \ \left\{Y^{i\beta}, Y^{\dagger j}_\alpha\right\} \qquad (J = 2)$	$(2,1)_0$

TABLE XII. $SU(4)\times SU(2)\times U(1)$ Identities for $\Delta S\neq 0$. The second column gives the $(J,I)_S$ transformation properties of the operator identities.

$\left(\frac{\tau^a}{2}\right)^\alpha_\beta \left\{I^a,t^\beta\right\}=\left\{Y^{i\alpha},J^i_{ud}\right\}$	$(0,1/2)_1$
$\left\{Y^{i\alpha},J^i_s\right\}=\frac{1}{4}\left\{N_s,t^\alpha\right\}+\frac{1}{2}t^\alpha$	$(0,1/2)_1$
$\left(\frac{\tau^a}{2}\right)^\alpha_\beta \left\{Y^{i\beta},G^{ia}\right\}=\frac{3}{8}\ (N+2)\,t^\alpha-\frac{3}{16}\left\{N_s,t^\alpha\right\}-\frac{1}{2}\ \left\{Y^{i\alpha},J^i_{ud}\right\}$	$(0,1/2)_1$
$\left\{Y^{i\alpha},G^{ia}\right\}=\frac{1}{4}\left\{t^\alpha,I^a\right\}\qquad(I=3/2)$	$(0,3/2)_1$
$i\epsilon^{ijk}\left\{Y^{i\alpha},J^j_s\right\}=-\frac{1}{2}\left\{t^\alpha,J^k_s\right\}+\frac{1}{2}\left\{N_s,Y^{k\alpha}\right\}+Y^{k\alpha}$	$(1,1/2)_1$
$\left(\frac{\tau^a}{2}\right)^\alpha_\beta \left\{t^\beta,G^{ka}\right\}=\frac{1}{2}i\epsilon^{ijk}\left\{Y^{i\alpha},J^j_{ud}\right\}+\frac{1}{2}(N+2)\,Y^{k\alpha}-\frac{1}{4}\left\{N_s,Y^{k\alpha}\right\}$	$(1,1/2)_1$
$i\epsilon^{ijk}\left(\frac{\tau^a}{2}\right)^\alpha_\beta \left\{Y^{i\beta},G^{ja}\right\}=\frac{1}{4}\left\{t^\alpha,J^k_{ud}\right\}-\frac{1}{4}i\epsilon^{ijk}\left\{Y^{i\alpha},J^j_{ud}\right\}-\frac{1}{2}(N+2)\,Y^{k\alpha}+\frac{1}{4}\left\{N_s,Y^{k\alpha}\right\}$	$(1,1/2)_1$
$\left(\frac{\tau^a}{2}\right)^\alpha_\beta \left\{Y^{k\beta},I^a\right\}=\frac{1}{4}\left\{t^\alpha,J^k_{ud}\right\}-\frac{1}{2}i\epsilon^{ijk}\left\{Y^{i\alpha},J^j_{ud}\right\}$	$(1,1/2)_1$
$i\epsilon^{ijk}\ \left\{Y^{i\alpha},G^{ja}\right\}=\frac{1}{2}\left\{t^\alpha,G^{ka}\right\}-\frac{1}{2}\left\{Y^{k\alpha},I^a\right\}\qquad(I=3/2)$	$(1,3/2)_1$
$\left(\frac{\tau^a}{2}\right)^\alpha_\beta \left\{Y^{i\beta},G^{ja}\right\}=\frac{1}{4}\left\{Y^{i\alpha},J^j_{ud}\right\}\qquad(J=2)$	$(2,1/2)_1$
$\left\{Y^{i\alpha},Y^{i\beta}\right\}=\frac{1}{4}\left\{t^\alpha,t^\beta\right\}\qquad(I=1)$	$(0,1)_2$
$\epsilon^{ijk}\epsilon_{\alpha\beta}\ \left\{Y^{i\alpha},Y^{j\beta}\right\}=i\epsilon_{\alpha\beta}\left\{t^\alpha,Y^{k\beta}\right\}$	$(1,0)_2$

TABLE XIII. Operator reduction for $SU(2)\times U(1)$ flavor symmetry. The second column gives the allowed $(J,I)_S$ quantum numbers for the operators in column one. The third column gives the operators left after eliminating all redundant linear combinations using the identities in Tables XI and XII. Operator products not shown, such as $\left\{J^i_{ud},J^j_s\right\}$, do not have any linear combinations which can be eliminated.

$\left\{G^{ia},G^{jb}\right\}$	$(0,0)_0\ (1,1)_0\ (0,2)_0\ (2,0)_0\ (2,2)_0$	$(2,2)_0$
$\left\{I^a,I^b\right\}$	$(0,0)_0\ (0,2)_0$	$(0,2)_0$
$\left\{I^a,G^{ib}\right\}$	$(1,0)_0\ (1,1)_0\ (1,2)_0$	$(1,2)_0$
$\left\{J^i_{ud},G^{ja}\right\}$	$(0,1)_0\ (1,1)_0\ (2,1)_0$	$(1,1)_0\ (2,1)_0$
$\left\{J^i_s,J^j_s\right\}$	$(0,0)_0\ (2,0)_0$	$(2,0)_0$
$\left\{Y^{i\alpha},Y^{\dagger j}_\beta\right\}$	$(0,0)_0\ (0,1)_0\ (1,0)_0\ (1,1)_0\ (2,0)_0\ (2,1)_0$	
$\left\{Y^{i\alpha},t^\dagger_\beta\right\}$	$(1,0)_0\ (1,1)_0$	
$\left\{t^\alpha,Y^{\dagger i}_\beta\right\}$	$(1,0)_0\ (1,1)_0$	
$\left\{t^\alpha,t^\dagger_\beta\right\}$	$(0,0)_0\ (0,1)_0$	
$\left\{I^a,t^\alpha\right\}$	$(0,1/2)_1\ (0,3/2)_1$	$(0,3/2)_1$
$\left\{J^i_s,Y^{j\alpha}\right\}$	$(0,1/2)_1\ (1,1/2)_1\ (2,1/2)_1$	$(2,1/2)_1$
$\left\{I^a,Y^{i\alpha}\right\}$	$(1,1/2)_1\ (1,3/2)_1$	$(1,3/2)_1$
$\left\{G^{ia},t^\alpha\right\}$	$(1,1/2)_1\ (1,3/2)_1$	$(1,3/2)_1$
$\left\{G^{ia},Y^{j\alpha}\right\}$	$(0,1/2)_1\ (0,3/2)_1\ (1,1/2)_1\ (1,3/2)_1\ (2,1/2)_1\ (2,3/2)_1$	$(2,3/2)_1$
$\left\{Y^{i\alpha},Y^{j\beta}\right\}$	$(0,1)_2\ (1,0)_2\ (2,1)_2$	$(2,1)_2$

TABLE XIV. $SU(Q)$ Representations

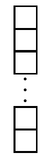
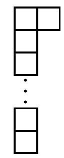
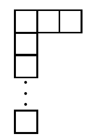
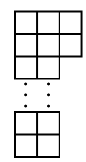
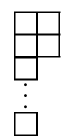
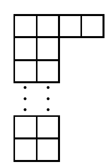
Rep	Dimension	Casimir	Dynkin Label	Young Tableau
1		0	$[0,0,0,0,\dots,0,0,0]$	
$\square$	Q	$\frac{Q^2-1}{2Q}$	$[1,0,0,0,\dots,0,0,0]$	
$\overline{\square}$	Q	$\frac{Q^2-1}{2Q}$	$[0,0,0,0,\dots,0,0,1]$	
adj	$\left(Q^2-1\right)$	Q	$[1,0,0,0,\dots,0,0,1]$	
$\bar{a}s$	$\frac{1}{4}\left(Q^2-1\right)\left(Q^2-4\right)$	$2Q$	$[2,0,0,0,\dots,0,1,0]$	
$\bar{s}a$	$\frac{1}{4}\left(Q^2-1\right)\left(Q^2-4\right)$	$2Q$	$[0,1,0,0,\dots,0,0,2]$	
$\bar{a}a$	$\frac{1}{4}Q^2\left(Q+1\right)\left(Q-3\right)$	$2\left(Q-1\right)$	$[0,1,0,0,\dots,0,1,0]$	
$\bar{s}s$	$\frac{1}{4}Q^2\left(Q-1\right)\left(Q+3\right)$	$2\left(Q+1\right)$	$[2,0,0,0,\dots,0,0,2]$	

TABLE XV. $\left(adj \otimes adj\right)_A$

	$SU(Q)$	$SU(6)$	$SU(4)$
$\left(adj \otimes adj\right)_A$	$\left([1,0,0,0,\dots,0,0,1]^2\right)_A$	$\left([1,0,0,0,1]^2\right)_A$	$\left([1,0,1]^2\right)_A$
adj $\bar{a}s$ $\bar{s}a$	$[1,0,0,0,\dots,0,0,1]$ $[2,0,0,0,\dots,0,1,0]$ $[0,1,0,0,\dots,0,0,2]$	$[1,0,0,0,1]$ $[2,0,0,1,0]$ $[0,1,0,0,2]$	$[1,0,1]$ $[2,1,0]$ $[0,1,2]$

TABLE XVI. S-wave hyperon non-leptonic decay amplitudes s. Experimental amplitudes are given in the second column. The third column is the one-parameter $SU(3)$ symmetric fit keeping only the leading operator in the $1/N_c$ expansion. The fourth column is the two-parameter $SU(3)$ symmetric fit including the leading operator and the subleading $1/N_c$ correction. A soft-pion theorem has been used in the calculation.

Decay	Expt	1	$1/N_c$
$\Sigma^+ \to n\pi^+$	0.06 ± 0.01	0.0	0.0
$\Sigma^+ \to p\pi^0$	-1.43 ± 0.05	-1.00	-1.35
$\Sigma^- \to n\pi^-$	1.88 ± 0.01	1.41	1.90
$\Lambda^0 \to p\pi^-$	1.42 ± 0.01	1.73	1.44
$\Lambda^0 \to n\pi^0$	-1.04 ± 0.01	-1.22	-1.02
$\Xi^- \to \Lambda\pi^-$	-1.98 ± 0.01	-1.73	-1.88
$\Xi^0 \to \Lambda\pi^0$	1.52 ± 0.02	1.22	1.33

FIG. 1. $SU(2F)$ representation for ground-state baryons. The Young tableau has N_c boxes.

FIG. 2. Weight diagram for the $SU(3)$ flavor representation of the spin-$\frac{1}{2}$ baryons. The top of the weight diagram has baryons with zero strange quarks. The long side of the weight diagram contains $\frac{1}{2}(N_c + 1)$ weights. The numbers denote the multiplicity of the weights.

FIG. 3. Weight diagram for the $SU(3)$ flavor representation of the spin-$\frac{3}{2}$ baryons. The top of the weight diagram has baryons with zero strange quarks. The long side of the weight diagram contains $\frac{1}{2}(N_c - 1)$ weights. The numbers denote the multiplicity of the weights.

FIG. 4. Feynman diagrams depicting the insertion of a one-quark QCD operator on the N_c quark lines of the baryon. Graphs (b) contain additional planar gluons, and are of the same order as (a).

FIG. 5. The hyperfine mass splittings within a baryon tower. Splittings at the bottom of the tower are of order $1/N_c$, whereas splittings at the top of the tower are of order 1. There are $(N_c + 1)/2$ energy levels in the flavor symmetry limit.

FIG. 6. Diagrams contributing to the quark mass dependence of the axial current matrix element. The axial current is denoted by $\otimes$, and the symmetry breaking Hamiltonian by a solid square. Diagrams such as (a) do not break the flavor symmetry. Flavor symmetry breaking arises from diagrams such as (b).

Figure 1

1 1
1 2 1
1 2 2 1
1 2 2 2 1
1 2 2 2 2 1
1 2 2 2 2 2 1
1 2 2 2 2 2 2 1
1 2 2 2 2 2 2 2 1
1 2 2 2 2 2 2 2 2 1
1 1 1 1 1 1 1 1 1

Figure 2

1 1 1 1
1 2 2 2 1
1 2 3 3 2 1
1 2 3 4 3 2 1
1 2 3 4 4 3 2 1
1 2 3 4 4 4 3 2 1
1 2 3 4 4 4 4 3 2 1
1 2 3 3 3 3 3 2 1
1 2 2 2 2 2 2 1
1 1 1 1 1 1 1

Figure 3

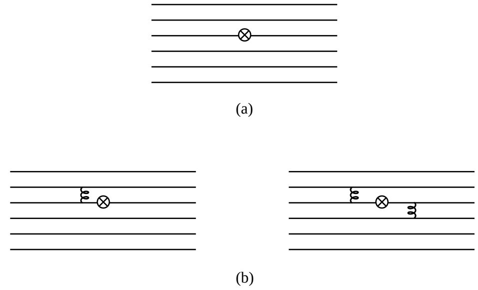


Figure 4

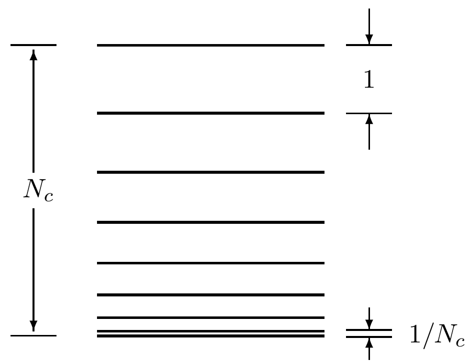


Figure 5

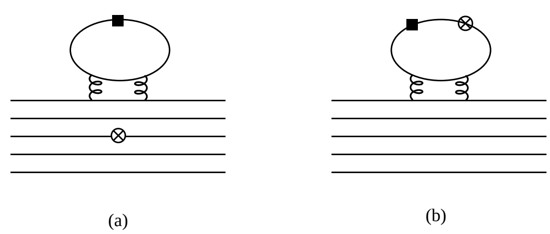


Figure 6